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978-0-521-88245-3 - Classical and Multilinear Harmonic Analysis: Volume I

Camil Muscalu and Wilhelm Schlag

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## Classical and Multilinear Harmonic Analysis

This two-volume text in harmonic analysis introduces a wealth of analytical results and techniques. It is largely self-contained and is intended for graduates and researchers in pure and applied analysis. Numerous exercises and problems make the text suitable for self-study and the classroom alike.

This first volume starts with classical one-dimensional topics: Fourier series; harmonic functions; Hilbert transforms. Then the higher-dimensional Calderón–Zygmund and Littlewood–Paley theories are developed. Probabilistic methods and their applications are discussed, as are applications of harmonic analysis to partial differential equations. The volume concludes with an introduction to the Weyl calculus.

The second volume goes beyond the classical to the highly contemporary and focuses on multilinear aspects of harmonic analysis: the bilinear Hilbert transform; Coifman–Meyer theory; Carleson’s resolution of the Lusin conjecture; Calderón’s commutators and the Cauchy integral on Lipschitz curves. The material in this volume has not been collected previously in book form.

**Camil Muscalu** is Associate Professor in the Department of Mathematics at Cornell University.

**Wilhelm Schlag** is Professor in the Department of Mathematics at the University of Chicago.

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# Classical and Multilinear Harmonic Analysis

Volume I

CAMIL MUSCALU  
*Cornell University*

WILHELM SCHLAG  
*University of Chicago*



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## Preface

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Harmonic analysis is an old subject. It originated with the ideas of Fourier in the early nineteenth century (which were preceded by work of Euler, Bernoulli, and others). These ideas were revolutionary at the time and could not be understood by means of the mathematics available to Fourier and his contemporaries. However, it was clear even then that the idea of representing any function as a superposition of elementary harmonics (sine and cosine) from an arithmetic sequence of frequencies had important applications to the partial differential equations of physics that were being investigated at the time, such as the heat and wave equations. In fact, it was precisely the desire to solve these equations that led to this bold idea in the first place.

Research into the precise mathematical meaning of such Fourier series consumed the efforts of many mathematicians during the entire nineteenth century as well as much of the twentieth century. Many ideas that took their beginnings and motivations from Fourier series research became disciplines in their own right. Set theory (Cantor) and measure theory (Lebesgue) are clear examples, but others, such as functional analysis (Hilbert and Banach spaces), the spectral theory of operators, and the theory of compact and locally compact groups and their representations, all exhibit clear and immediate connections with Fourier series and integrals. Furthermore, soon after Fourier proposed representing every function on a compact interval as a trigonometric series, his idea was generalized by Sturm and Liouville to expansions with respect to the eigenfunctions of very general second-order differential operators subject to natural boundary conditions – a groundbreaking result in its own right.

Not surprisingly harmonic analysis is therefore a vast discipline of mathematics, which continues to be a vibrant research area to this day. In addition, over the past 60 years Euclidean harmonic analysis, as represented by the schools associated with A. Calderón and A. Zygmund at the University of Chicago as

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well as these associated with C. Fefferman and E. Stein at Princeton University, has been inextricably linked with partial differential equations (PDEs). While applications to the theory of elliptic PDEs and pseudodifferential operators were a driving force in the development of the Calderón–Zygmund school from the very beginning, the past 25 years have also seen an influx of harmonic analysis techniques to the theory of nonlinear dispersive equations such as the Schrödinger and wave equations. These developments continue to this day.

The basic “divide and conquer” idea of harmonic analysis can be stated as follows: that we should study those classes of functions that arise in interesting contexts (for example, as solutions of differential equations; as measured data; as audio or video signals on DVDs, CDs, or possibly transmitted across glass fiber cables or great distances such as that between Mars and Earth; as samples of random processes) by breaking them into basic constituent parts and that (a) these basic parts are both as simple as possible and amenable to study and (b) ideally, reflect some structure inherent to the problem at hand.

In a classical manifestation these basic parts are given by the standing waves used in Fourier series, but over the past 30 years wavelets (as well as curvelets and ringlets) have revolutionized applied harmonic analysis, especially as used in image processing.

This fundamental idea is ubiquitous in science and engineering. Examples where it arises and is put to use include the following: all types of medical imaging (such as magnetic resonance imaging (MRI), computed tomography, ultrasound, echocardiography, and positron emission tomography (PET)); signal processing, especially through the methods used in compressed sensing; and inverse problems such as those that arise in remote sensing, medical imaging, geophysics, and oil exploration. In addition, advances in electrical engineering – and with it essentially the whole of modern industry as we know it – have only been possible through the systematic use and implementation of mathematics, often in the form of Fourier analysis and its ramifications.

To go even further, Nature appears to carry in herself the blueprint of the *basic harmonics*. Indeed, electrons and other elementary particles are understood as *spherical waves*, and the discrete energy levels so characteristic of quantum mechanics are dictated by the necessity of fitting such a standing wave onto a two-dimensional spherical surface. String theory takes this concept to an entirely different level of abstraction by reducing *everything* to the vibration of tiny strings.

Harmonic analysis has the advantage, over other subjects in mathematics, that it has never been completely isolated or divorced from applications; rather, a significant part of it has been steadily guided and inspired by them. For example, the study of the Cauchy kernel on Lipschitz curves might have arisen

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as a seemingly academic exercise and a rather mindless generalization but for the fact that there are so many important problems in materials science where Nature produces just such non-smooth boundaries.

Conceptual developments in harmonic analysis are at the center of many important scientific and technological advances. It is rather remarkable that wavelets are actually used in the JPEG 2000 standard. The technical details will be of interest mainly to specialists, but the conceptual framework has a much greater reach and perhaps significance. Theorems have hypotheses that may not be exactly satisfied in many real physical situations; in fact, engineering is not an exact science but rather one of good enough approximation. So, while theorems may not apply in a strict sense the thinking that went into them can still be extremely useful. It is precisely this thinking that our book wishes to present.

Classic monographs and textbooks in harmonic analysis include those by Stein [108], [110], and Stein and Weiss [112]; amongst the older literature there is the timeless encyclopedic work on Fourier series by Zygmund [131] and the more accessible introduction to Fourier analysis by Katznelson [65]. Various excellent, more specialized texts are also available, such as Folland [41], which focuses on phase-space analysis and the Weyl calculus, as well as Sogge [105], which covers oscillatory integrals. Wolff's lecture notes [128] can serve as an introduction to the ideas associated with the Kakeya problem. This is a more geometric, as well as combinatorial, aspect of harmonic analysis that is still rather poorly understood.

Our intention was not to compete with any of these well-known texts. Rather, our book is designed as a teaching tool, both in a traditional classroom setting as well as in the setting of independent study by an advanced undergraduate or beginning graduate student. In addition, the authors hope that it will also be useful for any mathematician or mathematically inclined scientist who wishes to acquire a working knowledge of select topics in (mostly Euclidean) Fourier analysis.

The two volumes of our book are different in both scope and character, although they should be perceived as forming a natural unit. In this first volume we introduce the reader to a broad array of results and techniques, starting from the beginnings of the field in Fourier series and then developing the theory along what the authors hope are natural avenues motivated by certain basic questions. The selection of the material in this volume is of course partly a reflection of the authors' tastes, but it also follows a specific purpose: to introduce the reader to sufficiently many topics in classical Fourier analysis, and in enough detail, to allow them to continue a guided study of more advanced material possibly leading to original research in analysis, pure or applied.

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All the material in this volume should be considered basic. It can be found in many texts but to the best of our knowledge not in a single place. The authors feel that this volume presents the course that they should have taken in graduate school, on the basis of hindsight. To what extent the entirety of this volume constitutes a reasonable course is up to the individual teacher to decide. It is more likely that selections will need to be made, and it is of course also to be expected that lecturers will wish to supplement the material with certain topics of their own choosing that are not covered here. However, the authors feel, particularly since this has been tested on individual students, that Vol. I could be covered by a beginning graduate student over the course of a year in independent, but guided, study. This would then culminate in some form of qualifying or “topic” exam, after which the student would be expected to begin independent research.

Particular emphasis has been placed on the inclusion of exercises and problems. The former are dispersed throughout the main body of the text and are for the most part an integral part of the theory. As a rule they are less difficult than the end-of-chapter problems. The latter serve to develop the theory further and to give the reader the opportunity to try his or her hand at the occasional hard problem. An old and commonplace principle, to which the authors adhere, is that any piece of mathematics can only be learned by active work, and this is reflected in our book. In addition, the authors have striven to emphasize intuition and ideas over both generality and technique, without, however, sacrificing rigor, elegance, or for that matter, relevance.

While Vol. I presents developments in harmonic analysis up to the mid to late 1980s, Vol. II picks up from there and focuses on more recent aspects. In comparison with the first volume the second is of necessity much more selective, and many topics of current interest could not be included. Examples of the many omitted areas that come to mind are the oscillatory integrals related to the Kakeya, restriction, and Bochner–Riesz conjectures, multilinear Strichartz estimates and geometric measure theory and its relations to combinatorics and number theory.

The selection of topics in Vol. II can roughly be described as phase-space oriented; it comprises material that either grew out of or is closely related to the David–Journé  $T(1)$  theorem, the Cauchy integral on Lipschitz curves, and the Calderón commutators on the one hand and the solution to Lusin’s problem by Carleson on the other hand. The latter work, also through Fefferman’s reworking of Carleson’s proof, greatly influenced the resolution of the bilinear Hilbert-transform boundedness problem by Lacey and Thiele in the mid 1990s. Generally speaking, Carleson’s work has had a profound influence on the more combinatorially oriented analysis of phase space that lies at the heart of Vol. II.

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To be more specific, Vol. II concentrates on paraproducts (which make an appearance in Vol. I), the bilinear Hilbert transform, Carleson's theorem, and Calderón commutators and the Cauchy integral on Lipschitz curves. In fact, the analysis of paraproducts can be seen as the most foundational material for Vol. II; paraproducts are developed on polydisks and as flag paraproducts in their own right. In this sense, Vol. II is more research oriented in character, as some results are quite recent and the organization and development of the material are in part original and specific to Vol. II.

In terms of presentation, in the second volume the authors have generally chosen not the shortest proofs but those that are most robust as well as (to their taste) most illuminating. For example, there are simpler ways of approaching the Coifman–Meyer theorem on paraproducts but these do not carry over to other contexts such as flag paraproducts. Much emphasis has been placed on motivation, and so the authors have included applications to PDEs, for example in the form of Strichartz estimates and their use in nonlinear equations in Vol. I and in the form of fractional Leibnitz rules with application to the KdV equation in Vol. II. The chapter entitled “Iterated Fourier series and physical reality” in Vol. II is entirely devoted to motivation and to an explanation of the larger framework in which much of that volume sits.

Throughout both volumes the authors have striven to emphasize intuition and ideas, and often figures have been used as an important part of an explanation or proof. This is particularly the case in Vol. II, which is more demanding in terms of technique and with often longer and more complicated proofs than those in Vol. I.

Volume II overlaps considerably with the recent research literature, especially with papers by Lacey and Thiele on the one hand and by the first author, Tao, and Thiele on the other hand. We shall not give an account of these papers here, since a discussion can be found in the end-of-chapter notes (we have avoided placing citations and references in the main body of the text, so as not to distract the reader).

We shall now comment briefly on the relation of Vol. II to classic textbooks in the area. The influential work by Coifman and Meyer [24] overlaps the first and fourth chapters of Vol. II. However, not only is the technical approach developed in [24] completely different from that in this book but it is also designed for a different purpose: it is less a textbook than a rapid review of many deep topics such as Hardy spaces on Lipschitz domains, Murai's proof of the Cauchy integral boundedness, and commutators and the Cauchy integral. Finally, [24] develops wavelets, which are an essential tool in many real-world applications but make no appearance in this book.

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Another well-known text that overlaps Vol. II contains Christ's CBMS lectures [21]. These are centered around the  $T(1)$  and  $T(b)$  theorems and their applications. While some of this material does make an appearance in our book, Vol. II does not use  $T(1)$  or  $T(b)$ .

We conclude this preface with a discussion of prerequisites. Essential is a grounding in basic analysis, beginning with multivariable calculus (including writing a hypersurface as a graph and the notion of Gaussian curvature on a hypersurface) going on to measure theory and integration and the functional analysis relevant to basic Hilbert spaces (orthogonal projections, bases, completeness) as well as to Banach spaces (weak and strong convergence, bases, Hahn–Banach, uniform boundedness), and finally the basics of complex analysis (including holomorphic functions and conformal maps). In Vol. I probability theory also makes an appearance, and it might be helpful if the reader has had some exposure to the notions of independence, expected value, variance, and distribution functions. The second volume requires very little more in terms of preparation other than a fairly mature understanding of the above topics. The authors do not recommend, however, that one should attempt Vol. II before Vol. I.

As is customary in analysis, interpolation is frequently used. To be more specific, we rely on both the Riesz–Thorin and Marcinkiewicz interpolation theorems. We state these facts at the end of the first chapter but omit the proofs, as they can readily be found in the texts of Stein and Weiss [112] and Katznelson [65]. Finally, in Chapter 11 of the present volume, on the restriction theorem, we also make use of Stein's generalization of the Riesz–Thorin theorem to analytic families of operators. Volume II uses multilinear interpolation, and the facts needed are collected in an appendix. A standard reference for interpolation theory, especially as it relates to Besov, Sobolev, and Lorentz spaces, is the text by Bergh and Löfström [7].

As we have already pointed out, many important topics are omitted from our two volumes. Apart from classical topics such as inner and outer functions, which we could not include in Vol. I owing to space considerations, an aspect of modern harmonic analysis that is not covered here is the vast area dealing with oscillatory integrals. This field touches upon several other areas including geometric measure theory, combinatorial geometry, and number theory and is relevant to nonlinear dispersive PDEs in several ways, such as through bilinear restriction estimates (see for example Wolff's paper [127] as well as his lecture notes [128]). This material would naturally comprise a third volume, which would need to present the research that has been done since the work of Sogge [105] and Stein [109].

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**How to use this volume.** Ideally, the reader should work through the chapters in order. A reader or class familiar with the Fourier transform in  $\mathbb{R}^d$  could start with Chapter 7, then move on to Chapter 8 and subsequently choose any of the remaining four chapters in this volume according to taste and time constraints. Chapters 7 through 11 constitute the backbone of real-variable harmonic analysis. Of those, Chapter 10 can be regarded as an optional extra; however, the authors feel that it is of importance and that students should be exposed to this material. As an application of the Logvinenko–Sereda theorem of Section 10.3, we prove the local solvability of constant coefficient PDEs (the Malgrange–Ehrenpreis theorem) in Section 10.4.

Another such outlier is Chapter 12, in which we introduce the reader to an area that is itself the subject of many books; see for example Taylor’s book on pseudodifferential operators [120] as well as Hörmander’s treatise [58]. The present authors decided to include a very brief account of this story, since it is an essential part of harmonic analysis and also since it originates in Calderón’s work as part of his investigations of singular integrals and the Cauchy problem for elliptic operators. In principle, Chapter 12 can be read separately by a mature reader who is familiar with Cotlar’s lemma from Chapter 9. In the writing of this chapter a difficult decision had to be made, namely which of the two main incarnations of pseudodifferential operators to use, that of Kohn–Nirenberg or that of Weyl. While the former is somewhat simpler technically, and therefore often used for elliptic PDEs, the older Weyl quantization is very natural owing to its symmetry and is the one that is normally used in the so-called semiclassical calculus. We therefore chose to follow the latter route; Kohn–Nirenberg pseudo-differential operators make only a very brief appearance in this text.

In Chapter 5 we introduce the reader to probability theory, which is also often omitted in a more traditional harmonic analysis presentation. However, the authors felt that the ideas developed in that chapter (which are very elementary for the most part) are an essential part of modern analysis and of mathematics in general. They appear in many different settings and should be in the toolbox of any working analyst, pure or applied. Chapter 6 contains several examples of how probabilistic thinking and results appear in harmonic analysis. Section 6.3 on Sidon sets can be omitted on first reading, as it is somewhat specialized (it contains, in particular, Rider’s theorem, which we prove there).

The first four chapters of this volume are intended for a reader who has had no or very little prior exposure to Fourier series and integrals, harmonic functions, and their conjugates. A basic introductory advanced-undergraduate or beginning-graduate course would cover the first three chapters, omitting

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Section 2.5, and would then move on to the first section of the fourth chapter (here, the material on locally compact Abelian groups could be omitted entirely, since it is used only in a non-Euclidean setting in the proof of Rider's theorem in Section 6.3, and the stationary phase method is used only in the final two chapters of this volume). After that, an instructor could then select any topics from Chapters 5–8 as desired, the most traditional choice being the Calderón–Zygmund, Mihlin, and Littlewood–Paley theorems. The last of these theorems, at least as presented here, does require a minimal knowledge of probability, namely in the form of Khinchine's inequality.

**Feedback.** The authors welcome comments on this book and ask that they be sent to [harmonic@math.uchicago.edu](mailto:harmonic@math.uchicago.edu).

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Camil Muscalu and Wilhelm Schlag

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Aspects of multilinear harmonic analysis related to transversality. Jonathan Bennett. Abstract. The purpose of this article is to survey certain aspects of multilinear harmonic analysis related to notions of transversality. Very recently Bourgain and Guth [22] have developed a method for applying transversal multilinear inequalities to classical curvature-related linear problems. This has provided a mechanism through which unexpected techniques may be brought to bear on difficult curvature-related problems in harmonic analysis and PDE; see [20], [21], [23], [4] and [36]. In particular, this is responsible for the current state-of-the-art on the restriction conjecture for the Fourier transform. We do not attempt to properly survey these applications in this article. This two-volume text in harmonic analysis introduces a wealth of analytical results and techniques. It is largely self-contained and is intended for graduates and researchers in pure and applied analysis. Numerous exercises and problems make the text suitable for self-study and the classroom alike. The second volume goes beyond the classical to the highly contemporary and focuses on multilinear aspects of harmonic analysis: the bilinear Hilbert transform; Coifman–Meyer theory; Carleson’s resolution of the Lusin conjecture; Calderón–Zygmund commutators and the Cauchy integral on Lipschitz curves. The material in this volume has not been collected previously in book form.