Logic and Quantum Physics

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Current research in Logic is no longer confined to the traditional study of logical consequence or valid inference. As can be witnessed by the range of topics covered in this special issue, the subject matter of logic encompasses several kinds of informational processes ranging from proofs and inferences to dialogues, observations, measurements, communication and computation. What interests us here is its application to quantum physics: how does logic handle informational processes such as observations and measurements of quantum systems? What are the basic logical principles fit to handle and reason about quantum physical processes? These are the central questions in this paper. It is my aim to provide the reader with some food for thought and to give some pointers to the literature that provide an easy access to this field of research.

In the next section I give a brief historical sketch of the origin of the quantum logic project. Next I will explain the theory of orthomodular lattices in section 2. Section 3 covers the syntax and semantics of traditional quantum logic. In section 4, I focus on the limits of quantum logic, dealing in particular with the implication problem. This paves the way to section 5 on modal quantum logic. I end with section 6 on dynamic quantum logic, giving the reader a taste of one of the latest new developments in the field.

1   The origin of the quantum logic project

J. von Neumann, whose name is well known in Mathematics, Computer Science and Game Theory is also known as the father of quantum logic. It is only a very short paragraph in chapter 3 of his book *Grundlagen der Quantenmechanik* [50] that forms the birth of quantum logic. In this passage, von Neumann introduced the idea of a logical calculus of physical properties. He argued that it is the relation between these properties on the one hand and the projection operators definable on a Hilbert space on the other

hand that should make it possible to obtain some sort of logical calculus. However, we had to wait till 1936 when von Neumann co-authored a paper with G. Birkhoff [12] to actually see a full analysis of this logical system. The aim of their quantum logic project in [12] was to discover the logical structure of quantum mechanics. Certainly their project wouldn't have triggered so much interest in the physics, mathematics and philosophy communities, were it not for the surprising conclusion they obtained. Namely, what they argued for is that the logical structure of quantum mechanics does not conform to classical logic. Indeed, Birkhoff and von Neumann explained that the logic of experimental propositions as a calculus is different from classical logic but nonetheless "resembles the usual calculus of propositions with respect to *and, or* and *not*" [12]. They went on to study this system and analyze its difference with classical logic further, indicating that it are not the properties of negation but rather those of distributivity (of conjunction over disjunction) that form the "weakest link in the algebra of logic". In their paper the distributive law had to make place for a weaker version, the so-called modular law. A later paper by K. Husimi [31] actually showed that a further weakening of this law to the orthomodular law (see section 2) is necessary in the axiomatization of the logic of projection operators on a Hilbert space.[1]

While the original quantum logic project started from an axiomatics directly based on an underlying Hilbert space structure, we notice a shift in focus from the 1950's on. In particular, the aim was now to search for *logical conditions* without a priori starting from the Hilbert space structure. With this task set forward, the quantum logic project fully lifted of the ground. A small new community of mathematicians and logicians got involved in the project, trying to obtain *a representation theorem* with respect to the Hilbert space model. This revival in the field originated with the work of G. Mackey in [35, 36]. Mackey searched for a list of transparant and physically plausible axioms or assumptions from which the Hilbert space model could ideally be deduced. The programme of Mackey was later further developed and extended by C. Piron in his PhD thesis in 1964 [42] and subsequent work. Rather than taking the Hilbert space model of quantum physics for granted, Piron's aim was to "*justify the use of Hilbert space*" [43]. And that is exactly what Piron's celebrated representation theorem gives us: an axiomatic system that can be represented as the logic of projection operators on a generalized Hilbert space. Piron's theorem was later improved by M.

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[1]As remarked by D. Foulis in [22]: L. Loomis [34] and S. Maeda [37] independently rediscovered Husimi's orthomodular law in 1955.

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Solèr and R. Mayet [46, 38], who added one more axiom to the list to obtain a representation theorem with respect to the standard infinite-dimensional Hilbert spaces. It is fair to say that these results still from the highlight of our research field today.

Further work in the area went in two main directions: 1) a recasting of the Piron-Solèr-Mayet axioms in a propositional logic, and 2) the quest of a similar representation theorem with respect to tensored Hilbert spaces describing compound physical systems. Concerning the first direction, there were several reasons why a recasting of those axioms in a propositional setting would be desirable. First of all, the axiomatization of Piron-Solèr-Mayet is not first-order. This means that some of the Piron-Solèr-Mayet axioms cannot be stated in the first-order language of orthomodular lattices.[2] Secondly, some of these axioms have a rather un-intuitive character. In particular the orthomodular law is of a rather artificial nature and has left researchers wondering about its possible physical meaning. Note that the quest for an intuitive physical explanation for these axioms goes back to Birkhoff and von Neumann who ended their 1936 paper exactly with the question what a simple and plausible physical motivation for the modular law would be. One of the interesting lines of research in this direction is the development of modal quantum logic which I will briefly sketch in section 5, followed by an outline of the research programme on dynamic quantum logic in section 6.

Concerning the second direction, the community obtained important impossibility results pointing to a weakness in the original quantum logic project. While the orthomodular lattice approach proved successful to describe single quantum systems, it faced problems when trying to describe compound systems consisting of subsystems that can exhibit quantum entanglement. In quantum mechanics, such compound systems are represented via a tensor product of the underlying Hilbert spaces for each subsystem. Hence it would have been natural to find a general lattice-theoretic analogue of the tensor product as an operation on lattices that satisfies a given set of natural conditions. However, impossibility results in [1, 47] show that such an operation on orthomodular lattices (or posets) cannot exist. As a consequence, the focus in the quantum logic community shifted towards more general and abstract structures trying to accommodate the construction of a tensor product. The developments went into many directions, from the study of orthoalgebras, effect algebras to quantales. In even more recent

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[2]For instance the axiom of Solèr and Mayet requires quantification over higher-level objects, such as automorphisms of the given lattice.

work [27, 2, 3, 18, 6, 7, 5], the authors connect the logical foundations of quantum physics directly to fundamental issues in logic and theoretical computer science, and in particular to theories of distributed computation and quantum information.

Further details on the history of the development of the quantum logic project can be found in [19, 14, 22, 20].

2 Orthomodular Lattices

Orthomodular lattices (also called OML's) are used to capture the structure of the properties of a quantum physical system. So let us start from a given physical system and represent its state by a "ray" (or a one-dimensional subspace) in a given Hilbert space H as follows:

$$s = \{\lambda.x : \lambda \in C\}$$

where $\lambda \in C$ denotes an arbitrary complex scalar and $x \in H \backslash \{0\}$ a vector in the given Hilbert space H.[3] We call Σ the state space, which is given by the set of all "states" (rays) of a Hilbert space H.

A physical system also has properties such as e.g. "observable X takes value Y" where X can be the position of the system, its momentum or some other observable, and Y an element of a given result structure. In Logic, a property can be represented via the set of states, meaning the states in which the system exhibits that property. Hence, a "logical property" of a quantum system corresponds to what is called a "logical proposition", i.e. a set $S \subseteq \Sigma$ indicating the states in which the corresponding proposition is true. Note that not every set of vectors of H corresponds to a logical property, but only sets that are closed under multiplication by complex scalars; in other words, sets that are *unions of one-dimensional subspaces*.

In the quantum logic literature, one usually concentrates only on properties that can in principle be tested by a measurement. These are called *testable properties* or *experimental propositions*. In our Hilbert space H, the testable properties correspond to the *closed linear subspaces* of H: Given a subspace $M \subseteq H$, let us first denote the orthogonal subspace by $\sim M = \{y \in H : \forall x \in M \langle x, y \rangle = 0\}$ where $\langle x, y \rangle$ is the notation for the inner product. Next, $M \subseteq H$ is called a *closed linear subspace* of H if and only if it is biorthogonally closed, i.e. $\sim \sim M = M$.

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[3]Contrary to the tradition in quantum logic, the custom in quantum information theory is to work with states as vectors of a Hilbert space and not with states as one-dimensional subspaces.

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Let us call $\mathcal{L}(H)$ the family of all closed linear subspaces of H. And let us denote by $\mathcal{L}$ the family of all subsets $P = \{s \in \Sigma : s \subseteq W\}$ of Σ corresponding to the closed linear subspaces $W \subseteq H$. Hence $\mathcal{L}$ represents the family of all testable properties on a more abstract level.

The testable properties of a physical system are ordered in a certain way. To quote Birkhoff and von Neumann: "The first postulate concerning propositional calculi is that the physical qualities [properties] attributable to any physical system form a partially ordered system" [12]. The partial order relation on $\mathcal{L}$ is given by the set theoretical inclusion $\subseteq$ (which is reflexive, antisymmetric and transitive). The order on $(\mathcal{L}(H), \subseteq)$ is the inclusion between closed subspaces.

One then obtains a *lattice* of properties by considering two operations on the elements $M, N \in \mathcal{L}$: 1) the *meet* (or the greatest lower bound) $M \cap N$ and 2) the *join* (or the least upper bound): $M \sqcup N := \sim\sim (M \cup N)$. Note that $M \sqcup N$ is the subspace generated by the union $M \cup N$ and it consists of all *"superpositions"* of states in M and N. It is easy to see that the join $\sqcup$ is different from the union (or classical disjunction). Let us denote the smallest element of these lattices by 0 and the largest element by 1.

The family of closed linear subspaces $\mathcal{L}(H)$ comes equipped with the map $\sim: \mathcal{L}(H) \to \mathcal{L}(H), M \mapsto\sim M$, which induces a map $\sim: \mathcal{L} \to \mathcal{L}$ on testable properties called the *orthocomplementation* and satisfies the following conditions for every $P, Q \in \mathcal{L}$:

$$P \cap \sim P = 0$$

$$P \sqcup \sim P = 1$$

$$P =\sim\sim P$$

$$P \subseteq Q \Rightarrow \sim Q \subseteq\sim P$$

The lattice $(\mathcal{L}, \subseteq, \cap, \sqcup, \sim, 0, 1)$ is said to be an orthomodular lattice if in addition it satisfies the weak modular law (or also known as the orthomodular law):

For all $P, Q \in \mathcal{L} : P \subseteq Q$ implies $Q = P \sqcup (Q \cap \sim P)$

Weak Modularity is a weaker condition than distributivity of $\sqcup$ over $\cap$ (i.e. $P \sqcup (Q \cap R) = (P \sqcup Q) \cap (P \sqcup R)$ and $P \cap (Q \sqcup R) = (P \cap Q) \sqcup (P \cap R)$). As stressed by Birkhoff and von Neumann in [12]: "Distributivity is a law in classical, not quantum mechanics". Via the given definitions, it is easy to see that *distributivity* implies *weak modularity*, but not the other way

around. The Hasse diagram of an orthomodular lattice in Fig.1 shows how distributivity fails: $B \cap (\sim R \sqcup \sim B) = B$ while $(B \cap \sim R) \sqcup (B \cap \sim B) = 0$.
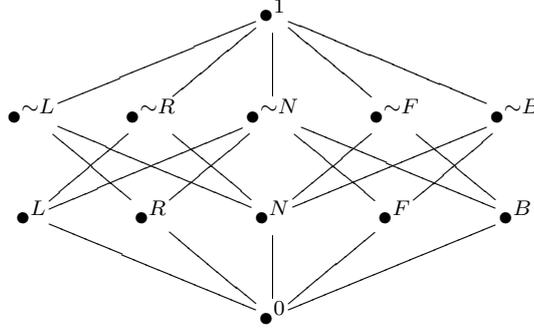

Fig. 1 Hasse Diagram of an orthomodular lattice, see [22].

To recapitulate, the structure of the *testable properties* of a physical system is given by an (at least) *orthomodular lattice* $(\mathcal{L}, \subseteq, \cap, \sqcup, \sim, 0, 1)$ with $\mathcal{L} \subseteq \mathcal{P}(\Sigma)$. It differs as such from the structure of the *logical properties* of a given physical system. The later is based on the powerset of the state space $\mathcal{P}(\Sigma)$, which is ordered by inclusion and equipped with the standard set operations (intersection, union, complement) so that it forms a *Boolean algebra*.

Looking back to the historical remarks in the previous section: a Lattice-theoretic axiomatization that is Hilbert-Complete does exist for quantum logic. The mentioned representation theorem of Piron-Solèr-Mayet shows that we can indeed equip an abstract property lattice $\mathcal{L}$ (not necessarily based on an underlying Hilbert space) with axioms that ensure $\mathcal{L}$ to be equivalent to the lattice of closed linear subspaces $\mathcal{L}(H)$ for some infinite dimensional H. The axioms are rather complicated and require $\mathcal{L}$ to be a complete, atomic, irreducible, orthomodular lattice (of a sufficient dimension) which satisfies Piron's covering law and the Solèr-Mayet condition. The covering law expresses that if $P \in \mathcal{L}$ and Q is an atom in $\mathcal{L}$ then $P \cap (\sim P \sqcup Q)$ is also an atom in $\mathcal{L}$. The Solèr-Mayet condition asks for the existence of a lattice automorphism $u : \mathcal{L} \to \mathcal{L}$, which has a set of fixed points of dimension ≥ 2, and which maps some element $P \in \mathcal{L}$ to some $u(P) \subset \neq P$. In terms of Hilbert spaces, the last condition is needed to ensure the existence of an infinite set of orthonormal vectors.

For more information about this lattice theoretic setting, I refer the interested reader in first instance to [33, 19, 51].

3 Quantum Logic, its syntax and semantics

Let us start by considering the Backus-Naur Form (or BNF) of the following language L, which is build up from a given set Ω of basic (elementary) formulas p:

$$\varphi \;\; ::= \;\; p \;\; | \;\; \varphi \wedge \varphi \;\; | \sim \varphi$$

 The BNF definition specifies that the well-formed formulae of L comprise only those symbols generated recursively from the basic formulas, the binary connective $\wedge$ (which is called the conjunction) and the unary connective $\sim$ (which is called the orthocomplement). One can use $\wedge$ and $\sim$ to define a connective for the "join" in our language L as follows: $\varphi \sqcup \psi := \sim (\sim \varphi \wedge \sim \psi)$. We write $\bot := p \wedge \sim p$ for the bottom element (or falsum) in L and $\top := \sim \bot$ for the top element (or verum). The behavior of these connectives is regulated by a proof theory and their interpretation is given via a semantics.

In the *Hilbert space semantics* the models are based on the structure of H. The interpretation of an elementary formula $|| \, p \, ||$ in this setting is given by a closed linear subspace of H, or similarly by a testable property of $\mathcal{L}$. The interpretation of the conjunction $|| \, \varphi \wedge \psi \, ||$ is given by the intersection $|| \, \varphi \, || \cap || \, \psi \, ||$ and the interpretation of the orthocomplement $|| \sim \varphi \, ||$ is given by $\sim || \, \varphi \, ||$. It is sufficient to give the interpretations for the basic connectives in our language as the other operators can be derived using the given abbreviations in L. Hence the interpretation of $|| \, \varphi \sqcup \psi \, ||$ is given by $|| \, \varphi \, || \sqcup || \, \psi \, ||$.

Turning to the proof system, the most popular methods of axiomatizing logical systems are probably the Gentzen style sequent calculi and natural deduction systems. In the history and development of ortho(modular)logic we encounter several kinds of axiomatizations with/without an implication connective. For natural deduction systems and Gentzen style sequent calculi I refer to [15, 19, 24, 25, 39, 40, 41, 48, 49]. In the tradition of Hilbert-style axiomatizations one may place the implication algebra of P.D. Finch, the work of R. Piziak [44] and further in this line also the work of I.D. Clark in [13] who uses negation and implication as his basic connectives.

I will focus further only on one of the earlier proof systems, namely R. Goldblatt's binary quantum logic [25]. In this setting we will work with a collection of ordered pairs of well formed formulae as an indicator for which formulas are inferable from which others. The aim here is to axiomatize directly the consequence relation $\vdash$ of the logic at hand. So $\psi \vdash \varphi$ stands for "φ is a *consequence of (or is derivable from)* ψ within quantum logic". First

I will focus in this setting on *minimal quantum logic* or so-called *orthologic*, which means that it lacks the proof rule for the orthomodular law:

(identity)	$\varphi \vdash \varphi$
(transitivity)	if $\varphi \vdash \psi$ and $\psi \vdash \gamma$ then $\varphi \vdash \gamma$
($\wedge$-elimination)	$\varphi \wedge \psi \vdash \varphi$
($\wedge$-elimination)	$\varphi \wedge \psi \vdash \psi$
($\wedge$-introduction)	$\varphi \vdash \psi$ and $\varphi \vdash \gamma$ then $\varphi \vdash \psi \wedge \gamma$
(weak double negation)	$\varphi \vdash \sim\sim \varphi$
(strong double negation)	$\sim\sim \varphi \vdash \varphi$
(contraposition)	if $\varphi \vdash \psi$ then $\sim \psi \vdash \sim \varphi$
(ex falso)	$\varphi \wedge \sim \varphi \vdash \psi$

To obtain an *orthomodular logic*, let us add the following rule for orthomodularity to the above proof system of orthologic:

$$\varphi \wedge (\sim \varphi \sqcup (\varphi \wedge \psi)) \vdash \psi$$

In case one would want to go beyond the binary setting to allow the presence of unrestricted contexts (sets of formulae), care has to be taken to avoid deriving the following $\sim \varphi, \varphi \sqcup \psi \vdash \psi$ (where the comma on the left is standardly interpreted as classical conjunction). Given the Hilbert space semantics it is easy to see that the failure of this quantum version of the disjunctive syllogism goes hand in hand with the failure of distributivity.

A relational semantics for minimal quantum logic (without the rule for orthomodularity) can be given, following the work of Goldblatt, Dishkant, Dalla Chiara, and others. This semantics is based on a relational structure $(\Sigma, \rightarrow)$ consisting of a set of states Σ and a binary relation $\rightarrow \subseteq \Sigma \times \Sigma$ which is reflexive ($s \rightarrow s$) and symmetric (if $s \rightarrow t$ then $t \rightarrow s$). This binary relation is also called the "accessibility relation" in the context of modal logic and the "non-orthogonality relation" or "similarity relation" in the context of quantum logic. The relational structure $(\Sigma, \rightarrow)$ is also known as a *frame* or *similarity space* (or in its dual version $(\Sigma, \not\rightarrow)$ as an orthoframe, orthogonality space or preclusivity space).

Given a relational structure $(\Sigma, \rightarrow)$, an orthogonality relation can be defined as follows: $s \perp t$ iff $s \not\rightarrow t$. This matches to the orthogonality relation on H, namely $s \perp t$ iff $\forall x \in s \, \forall y \in t \, \langle x, y \rangle = 0$. These relational structures or frames can be seen as generalizations of Hilbert spaces with (non)-orthogonality as the essential ingredient. Next we define the ortho-complement of a set $S \subseteq \Sigma$ as $\sim S := \{t \in \Sigma \mid t \perp s, \text{forall } s \in S\}$ and call $\sim\sim S = S$ biorthogonally closed.

A philosophical/physical justification for these relational structures can be found in the work of Dishkant. To quote from [21]: "Quantum Logic is a logic of slowly changing restorable facts". Indeed, one of the basic characteristics of quantum systems is that performing a measurement on them changes the ontic facts. This is well known in quantum theory, but let us remark with R. Feynman that making an observation affects the phenomena, "but the point is that this effect cannot be disregarded or minimized or decreased arbitrarily by rearranging the apparatus ... the disturbance is necessary for the consistency of the viewpoint" [23]. Dishkant uses this idea as a philosophical basis for interpreting the accessibility relation $s \rightarrow t$, according to him a transition from s to t "is connected with fulfilling an experiment" [21] very much in the sense of a quantum observation. This explains why quantum logic can be viewed as a logic of slowly changing restorable facts: the accessibility relation on Σ is reflexive because repeating the same observation does not change the state. It is symmetric because there are always (in theory) experiments that give us the original state back. But it is not transitive, which means that one cannot change the states very fast. For instance while $s \rightarrow t$ and $t \rightarrow w$ may be perfectly possible in case s and w are orthogonal states, but we then also have $s \nrightarrow w$ as no single observation can induce a direct transition from a given state into an orthogonal state.

As an example, consider again Fig. 1 where the states are given by the set of atoms $\{L, R, N, F, B\}$. The following orthogonality relation holds between these states: $L \perp R$ and $L \perp N$ (which are below $\sim L$ in the given order). Hence: $L \leftrightarrow L, F, B$ and $R \leftrightarrow R, F, B$ and $N \leftrightarrow N$. The relational structure that matches this example is given in the following drawing:
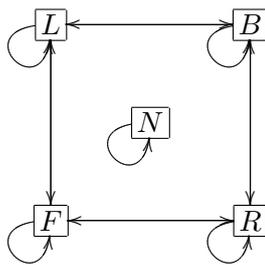

Fig.2 Relational Structure

A Kripke model $\boldsymbol{\Sigma} = (\Sigma, \rightarrow, || \, . \, ||)$ for minimal quantum logic is a relational structure $(\Sigma, \rightarrow)$ equipped with a valuation map $|| \, . \, ||: \Omega \rightarrow \mathcal{P}(\Sigma)$ that assigns biorthogonally closed subsets of Σ to elementary formulas. This

valuation can be extended to all formulas, defining an interpretation map as standard:

$$|| \perp || := \emptyset$$

$$|| \varphi \wedge \psi || := || \varphi || \cap || \psi ||$$

$$||\sim \varphi || := \{s \in \Sigma : \forall t(s \to t \Rightarrow t \notin || \varphi ||)\}$$

Let us also define the satisfaction relation as usual:

$$\Sigma, s \models_{QL} \varphi \quad \text{iff} \quad s \in || \varphi ||_{\Sigma}$$

And we denote the semantic consequence φ of T (where T is a set of formulas) as $T \models_{QL} \varphi$.

Similar as before, superpositions are captured using $\sqcup$, which differs from the classical disjunction as the following indicates:

$$s \in || \varphi || \text{ or } s \in || \psi || \Rightarrow s \in || \varphi \sqcup \psi || \text{ (but the converse fails)}$$

The orthocomplementation is expressed using $\sim$, which differs from the classical negation:

$$s \in ||\sim \varphi || \Rightarrow s \notin || \varphi || \text{ (but the converse fails)}$$

While these relational structures had been studied before (in particular in the work of D. Foulis and C. Randall), they were first used by Goldblatt in [25] to provide models for a propositional language. In his work, Goldblatt proves the soundness and completeness theorems for minimal quantum logic saying that a formula in minimal quantum logic is valid on all orthomodels if and only if it is a theorem of minimal quantum logic. All the given axioms of the proof theory, except for "orthomodularity", are indeed valid in the given Kripke semantics. However, when trying to extend this result to orthomodular logic (including the proof rule for orthomodularity), problems do arise. As shown in [26], there is no *first order condition* that can be imposed on the binary accessibility relation $\to$ that would be necessary and sufficient to validate the orthomodular law. Note that a first order condition is a condition expressible in the language of first order logic using the quantification over variables but no quantification over relations. e.g. Symmetry is a first order condition $\forall x, y(x \to y \Rightarrow y \to x)$ and transitivity is a first order condition $\forall x, y, z(x \to y \wedge y \to z \Rightarrow x \to z)$. As it turns out, it is not possible to capture the orthomodular law in this first order

language, it is said to be not elementary. This result is indeed important. To quote Goldblatt on it: "Most of the more important logics were shown to have their frames defined by first-order conditions on the relation R. [...] Thus the results of this paper indicate that the standard approach is unavailable for the logic of orthomodular lattices, as there is no first-order characterisation of orthomodularity for orthoframes. This is further evidence of the intractability of quantum logic. It is perhaps the first example of a natural and significant logic that leaves the usual methods defeated." [26].

The problem that there is no first-order characterisation of orthomodularity for orthoframes is actually just the tip of the iceberg if our aim is to establish a sound and complete axiomatization for a propositional quantum logic with respect to the Hilbert space semantics of $\mathcal{L}(H)$. In particular, if one wants to capture the so-called orthoarguesian law, one needs to go beyond the structure of orthomodular logic. The orthoarguesian law holds on $\mathcal{L}(H)$ but fails on the specific OML called G_{30} [17]:

$$a \subseteq b \sqcup \{(a \circ \sim b) \cap [(a \circ \sim c) \sqcup ((b \sqcup c) \cap ((a \circ \sim b) \sqcup (a \circ \sim c)))]\}$$

with $a \circ b := (a \sqcup \sim b) \cap b$. As I show in the last section of this paper, a dynamic logic setting will provide a way out of the here mentioned limits of the orthomodular logic approach. Further limitations such as the problem of implication are discussed in the next section.

4 Quantum logic and its limits: The Implication Problem.

As mentioned in the introduction, finding a *simple and plausible physical motivation* for the orthomodular law has been an important question in the quantum logic literature. I address this question here from a logical point of view and will show that the orthomodular law is closely related to the so-called quantum implication problem.

In classical logic one can define an implication operation as follows (using classical negation $\neg$ and classical disjunction $\vee$):

$$\varphi \Rightarrow \psi := \neg\varphi \vee \psi$$

In the classical propositional calculus, modus ponens holds for this implication, i.e. $\varphi \wedge (\varphi \to \psi) \vdash \psi$ and the deduction theorem of Herbrand (1930) is provable for it: If $\Gamma, \varphi \vdash \psi$ then $\Gamma \vdash \varphi \Rightarrow \psi$.

The deduction theorem together with modus ponens can be stated simply as the following *implicative rule* (or Galois adjunction):

$$\varphi \wedge \psi \vdash \gamma \quad \text{iff} \quad \psi \vdash \varphi \Rightarrow \gamma$$

In classical logic with a defined material implication one can easily prove that this implicative rule holds. On the other hand one can also use this rule as a *basic ingredient of the proof system*, to introduce an implication in classical or intuitionistic logic. Indeed, it is known due to a theorem of Skolem that a logic with a binary connective satisfying the implicative rule is necessarily distributive.

Hence it comes as no surprise that the mentioned implicative rule cannot be encountered in the case of quantum logic due to the mentioned failure of distributivity of $\wedge$ and $\sqcup$. This is exactly the origin of the implication problem. Right at the end of their paper [12], G. Birkhoff and J. von Neumann connect the failure of distributivity to this problem. I quote: "Our conclusion agrees perhaps more with those critiques of logic, which find most objectionable the assumption that $a' \cup b = \top$ implies $a \subset b$ (or dually, the assumption that $a \cap b' = \bot$ implies $b \supset a$ - the assumption that to deduce an absurdity from the conjunction of a and not b, justifies one in inferring that a implies b)."

It is natural to ask what kind of implications are possible in quantum logic if not the material implication? Let us first assume that any implication operation should satisfy the following *law of entailment*:

$$\varphi \vdash \psi \text{ iff } \vdash \varphi \Rightarrow \psi$$

Under this assumption, there are exactly five possible definitions for a binary implication in terms of meet and join operators, which do satisfy the law of entailment [33]:

$p_1(\varphi, \psi) = (\sim \varphi \wedge \psi) \sqcup (\sim \varphi \wedge \sim \psi) \sqcup (\varphi \wedge (\sim \varphi \sqcup \psi))$

$p_2(\varphi, \psi) = (\sim \varphi \wedge \psi) \sqcup (\varphi \wedge \psi) \sqcup ((\sim \varphi \sqcup \psi) \wedge \sim \psi)$

$p_3(\varphi, \psi) = \sim \varphi \sqcup (\varphi \wedge \psi)$

$p_4(\varphi, \psi) = \psi \sqcup (\sim \varphi \wedge \sim \psi)$

$p_5(\varphi, \psi) = (\sim \varphi \wedge \psi) \sqcup (\varphi \wedge \psi) \sqcup (\sim \varphi \wedge \sim \psi),$

Note that the classical material implication is not one them as it is well-known that there are counterexamples of orthomodular lattices for which we have $\vdash \sim \varphi \sqcup \psi$ but not $\varphi \vdash \psi$, see for instance the orthomodular lattice called M02 or *Chinese Lantern* in [33]. One of these five implications is of particular interest to us, namely p_3 the so-called Sasaki hook. Implication p_3 approaches the material implication of classical logic more closely than

the other candidates because it is *locally Boolean*, which means that on a classical sub-logic it will reduce to the classical material implication. In general however, the deduction theorem does fail for it. Hence it is not the case that: if $\Gamma, \varphi \vdash \psi$ then $\Gamma \vdash p_3(\varphi, \psi)$. The Sasaki Hook satisfies modus ponens $\varphi \wedge (p_3(\varphi, \psi)) \vdash \psi$, but as can easily be observed, this is nothing else than our proof rule for orthomodularity:

$$\varphi \wedge (\sim \varphi \sqcup (\varphi \wedge \psi)) \vdash \psi$$

The Sasaki hook has been the topic of further investigations. In [28, 29], G. Hardegree shows that the Sasaki hook - similar as the counterfactual or subjunctive conditional - in the non-Boolean orthomodular case, fails to satisfy strong transitivity $p_3(\varphi, \psi) \wedge p_3(\psi, \gamma) \vdash p_3(\varphi, \gamma)$, weakening $p_3(\varphi, \gamma) \vdash p_3((\varphi \wedge \psi, \gamma))$ as well as contraposition $p_3(\varphi, \psi) = p_3(\sim \psi, \sim \varphi)$. The failure of strong transitivity and contraposition of the Sasaki hook is pointed out explicitly by means of an example of a non-Boolean orthomodular lattice in [30]. Note that G. Hardegree has taken this discussion one step further by exploring the subjunctive/counterfactual nature of the Sasaki hook in relation to the operational characterization of quantum propositions, which in the work of C. Piron and J.M. Jauch is given in terms of underlying yes-no experiments - for details see [28, 29, 45].

5 Modal quantum logic

Given the above relational semantics in section 3, it is a small step to introduce a modal operator in our logical language. First I will present the modal logic $\mathcal{B}$ (called after Brouwer). Next I show how minimal quantum logic can be embedded in $\mathcal{B}$ such that all meaning is preserved, see [25, 16, 19].

Let us start with the language L^M of the modal logic $\mathcal{B}$, build up from a collection Ω of basic formulas p:

$$\varphi ::= \bot \mid p \mid \neg \varphi \mid \varphi \wedge \varphi \mid \Box \varphi$$

Our well formed formulas are obtained using classical negation $\neg$, classical conjunction $\wedge$ and a unary modal operator $\Box$.

The semantics for $\mathcal{B}$ is based on reflexive and symmetric Kripke frames $(\Sigma, \rightarrow)$. As standard, a Kripke model is a frame equipped with a valuation map $|| \, . \, ||$. This allows us to give an interpretation to the box operator:

$$|| \, \Box \varphi \, || := \{s \in \Sigma : \forall t (s \rightarrow t \Rightarrow t \in || \, \varphi \, ||)\}$$

Note that this is the standard interpretation for a $\Box$-modality in modal logic where $\Box\varphi$ often reads as "necessary φ". The classical dual of the $\Box$ is defined as $\Diamond\varphi := \neg\Box\neg\varphi$ and receives the following interpretation:

$$\| \Diamond\varphi \| = \{s \in \Sigma : \exists t, s \to t \text{ and } t \in \| \varphi \|\}$$

In modal logic, the $\Diamond$ operator standardly expresses a kind of "possibility" (as dual of $\Box$).

In the example of Fig.2, one can check that the following formulas are true or false at the state L in the given model:

$L \models_\mathcal{B} \Diamond\{B\}$; $L \models_\mathcal{B} \Diamond\{F\}$; $N \models_\mathcal{B} \Box\{N\}$; $L \not\models_\mathcal{B} \Box\{L\}$; $L \models_\mathcal{B} \Box\Diamond\{L\}$.

We can now use the logic $\mathcal{B}$ as a system that provides us with a modal way of looking at a non-classical logic such as the minimal quantum logic discussed in section 3. As stressed in [19], "From an intuitive point of view, it is useless to stress the interest of giving a modal interpretation of a non-classical logic; this permits to reinterpret with 'classical eyes' the behaviour and the meaning of a system of non-classical logical constants". To make the relation between $\mathcal{B}$ and minimal modal logic more precise, we follow [19] and define a translation τ of the language L of minimal quantum logic into the language L^M of $\mathcal{B}$:

$$\tau(p) = \Box\Diamond p$$

$$\tau(\sim \varphi) = \Box\neg\tau(\varphi)$$

$$\tau(\varphi \wedge \psi) = \tau(\varphi) \wedge \tau(\psi)$$

On the semantic side, the relation between both logics is straightforward. As proven in [19], one can show:

$$\models_{QL} \varphi \text{ iff } \models_\mathcal{B} \tau(\varphi) \text{ for any } \varphi \text{ of } L$$

$$T \models_{QL} \varphi \text{ iff } \tau(T) \models_\mathcal{B} \tau(\varphi) \text{ for any } \varphi \text{ and } T \text{ of } L$$

where T is a set of formulas and $\tau(T) = \{\tau(\varphi) \mid \varphi \in T\}$.

The given translation forms the basis of an embedding of minimal quantum logic into the Brouwer logic $\mathcal{B}$. One of the main advantages of this translation is that some of the nice meta-logical properties of $\mathcal{B}$ can be easily carried over to our discussion about minimal quantum logic. For instance, $\mathcal{B}$ has the finite model property. Using this result, one can show that minimal quantum logic has the finite model property and as it is finitely axiomatisable it is therefore also decidable. For a direct proof in terms of orthoframes of these metalogical properties, see [25].

6 Dynamic quantum logic

In the dynamic logic setting of [4], all the important qualitative properties of single quantum systems are cast as dynamic-logical properties. The earlier mentioned postulates of traditional quantum logic which had a rather technical nature (e.g. Non-distributivity, Orthomodularity, Piron's "Covering Law") can now be recovered as natural (although non-classical) properties of a quantum logical dynamics. On the technical side, an "abstract completeness result" for these axioms has been provided in [4], showing that *all* qualitative features of single quantum systems are captured by the axioms of dynamic quantum logic. In the remainder of this section, I will give a brief sketch of the main ideas of this setting following the work in [4, 5, 8].

First it was shown in [4] how Hilbert spaces can be structured as non-classical relational models of Propositional Dynamic Logic (PDL). These models are called Quantum Transition Systems (or QTS)

$$\mathbf{\Sigma} = (\Sigma, \xrightarrow{P?}, \xrightarrow{a})_{P \subseteq \Sigma, a \in \mathcal{A}}$$

they consist of a set of states Σ, come equipped with a family of transition relations between those states and satisfy a set of ten abstract semantical conditions. Similar as before, the states are meant to represent the *possible states of a physical system*. However what is new here is that the transition relations describe the changes of state induced by *possible actions* that may be performed on the system. There are two types of possible actions in play and for each one of them there are different corresponding labels on the accessibility relations. On the one hand there are relations labeled by properties (or propositions) $P \subseteq \Sigma$ and on the other hand there are relations labeled by elements a of a given set $\mathcal{A}$ of basic actions. As an example, consider the concrete QTS given by an infinite-dimensional Hilbert space H. Its states correspond to the one-dimensional subspaces, the relations of type $\xrightarrow{P?}$ correspond to quantum tests (or successful quantum yes/no-measurements) given by the projectors onto the closed linear subspace generated by the states corresponding to property P, and the relations of type $\xrightarrow{a}$ correspond to unitary actions or unitary linear maps a on H. The represenation theorem in [4] proves that every (abstract) QTS can be canonically embedded in the concrete QTS associated to an infinite-dimensional Hilbert space. This result has two important consequences: On the one hand it improves on the older complete axiomatizations of algebraic quantum logic and in particular on the result of Piron-Solér-Mayet mentioned before. Secondly, by using the QTS type of structures one can provide models for a propositional logical system and overcome the limitations discussed at the end of section 3.

The logical system at hand is called the *Logic of Quantum Actions* (LQA), it has the same language of (star-free) PDL. This is a language which consists of two layers: a layer of propositional *sentences* φ (expressing properties) and a layer of *programs* π (expressing actions) which are defined by mutual induction:

$$\begin{aligned} \varphi & \quad ::= \quad p| \quad \neg\varphi| \quad \varphi \wedge \psi| \quad [\pi]\varphi \\ \pi & \quad ::= \quad a| \quad \varphi?| \quad \pi \cup \pi| \quad \pi;\pi \end{aligned}$$

The variables p come from a given set of basic (elementary) propositions Ω and the basic action labels a come from the same set $\mathcal{A}$ as the ones in our QTS. We use $\neg$ to denote the *classical negation*, $\wedge$ the *classical conjunction*, $\varphi?$ the quantum test action, $\pi \cup \pi$ the non-deterministic choice of actions, $\pi;\pi$ the relational composition and $[\pi]\varphi$ for the construct that builds a new formulae from a given program π and formula φ. The semantics makes clear how we can interpret this. Our models are given by a QTS with a valuation map associating to each sentence φ a set of states $\|\varphi\| \subseteq \Sigma$ and to each program π a transition relation $\xrightarrow{\pi} \subseteq \Sigma \times \Sigma$. Quantum tests $\varphi?$ are interpreted using the transition relations labeled by $\|\varphi\|?$. We use the weakest precondition to give an interpretation to the dynamic modal operator:

$$\|[\pi]\varphi\| = \{s \in \Sigma : \forall t(s \xrightarrow{\pi} t \Rightarrow t \in \|\varphi\|)\}$$

It is the presence of classical negation $\neg$ in LQA which provides us with a major advantage over the older pure quantum structures, namely it ensures that this logic has strictly more expressive power than Quantum Logic. A similar reasoning applies to the modal logic $\mathcal{B}$, which is strictly more expressive than the minimal quantum logic. As stressed in [8], in the case of LQA we can consider the classical dual $<\varphi?>\psi := \neg[\varphi?]\neg\psi$ of the weakest precondition to express the *possibility* of actualizing a property ψ by a successful test of property φ. This allows us to express also the possibility of testing for φ via $\Diamond\varphi := <\varphi?>\top$.

This framework allows us to embed traditional (minimal) quantum logic into LQA, by means of the following translation tr of the language of orthologic into the language of LQA:

$$tr(p) = \Box\Diamond p$$

$$tr(\sim \varphi) = [tr(\varphi)?]\bot$$

$$tr(\varphi \wedge \psi) = tr(\varphi) \wedge tr(\psi)$$

One of the results in [8] indicates that *this translation is faithful on Hilbert spaces*: if $\mathbf{\Sigma}(H)$ is the concrete QTS induced by H, then for all states s and all quantum logic formulas φ we have

$$s \models_H \varphi \text{ iff } s \models_{\mathbf{\Sigma}(H)} tr(\varphi).$$

Observe that LQA as sketched above is a bivalent and Boolean propositional logic, in which the connectives satisfy all the classical laws of propositional logic. All the properties of quantum states and all non-classical features are captured via the dynamic-logic formulas. As the given translation indicates, we can recover the *orthocomplement* $\sim \varphi$ of a property as the *impossibility of a successful test* of φ. And the Sasaki hook can be defined as the weakest precondition of a quantum test action: $\varphi \xrightarrow{S} \psi := [\varphi?]\psi$. In other words, we obtain a *dynamic interpretation of the non-classical connectives of quantum logic*.

This work on dynamic quantum logic fits in the tradition of *the dynamic turn in Logic*, a recent trend in logical studies that has been pursued (mainly, but not exclusively) by the Dutch school in modal logic lead by J. van Benthem, see e.g. [10, 11]. What the basic setting in this section shows is that this dynamic turn in Logic includes the study of quantum informational processes. The given setting for single quantum systems can be extended to logics for complex "multi-partite" quantum systems, fit to reason about quantum communication and quantum computation on the basis of an informational-logical characterization of the notion of "quantum entanglement" [5, 6, 7, 9].

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Quantum Logic and Quantum Theory in a Game-Theoretic Perspective. Open Systems & Information Dynamics, Vol. 09, Issue. 03, p. 273.Â Kieu, Tien D. 2005. Quantum Principles and Mathematical Computability. International Journal of Theoretical Physics, Vol. 44, Issue. 7, p. 931. CrossRef. Google Scholar. Chiara, Maria Luisa Dalla Giuntini, Roberto Leporati, Alberto and Leporini, Roberto 2005. Qubit Semantics and Quantum Trees. International Journal of Theoretical Physics, Vol. 44, Issue. 7, p. 971. CrossRef. The 12th International Workshop on Quantum Physics and Logic will take place at the Department of Computer Science of the University of Oxford between Wednesday 15 and Friday 17 July, 2015. QPL is a workshop that brings together researchers working on mathematical foundations of quantum physics, quantum computing, spatio-temporal causal structures, and related areas such as computational linguistics. Of particular interest are topics that use logical tools, ordered algebraic and category-theoretic structures, formal languages, semantical methods and other computer science methods for the study Quantum Physics Overview. How Quantum Mechanics Explains the Invisible Universe. Share.Â In the realm of quantum physics, observing something actually influences the physical processes taking place. Light waves act like particles and particles act like waves (called wave particle duality). Matter can go from one spot to another without moving through the intervening space (called quantum tunnelling).