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THE ACQUISITION OF NUMERACY

Jenny Young-Loveridge ■ Department of Education Studies, University of Waikato

The New Zealand mathematics curriculum aims to provide students with the skills and understandings "to help them understand and play a responsible role in our democratic society".[1] The number system provides an important and powerful tool with which to do that. Understanding the number system is the key to teaching and learning mathematics. This article outlines a framework that shows how children think about the number system.

A NEW FRAMEWORK FOR THE ACQUISITION OF NUMERACY

Several researchers have developed models to explain how children's understanding about numbers develop as they progress from beginning to competent thinking.[2] However, each of these models has problems or limitations. For example, some focus on a very narrow domain of understanding, while others are overly complicated and difficult to apply quickly in a classroom context, and still others lack a clear rationale for progress to more advanced stages. For this reason, constructing a developmental framework to help teachers understand children's learning and thinking about the number system seemed important. Research in the United States and Australia has shown that teachers who were given this kind of framework were better able to help their students build on their mathematical thinking.[3,4] The framework described here is based predominantly on the work of Karen Fuson and Lauren Resnick in the United States, and integrates many important features of the other models.

The framework consists of four stages, each characterised by a major shift in ways of thinking about numbers (*see* figure 1). There is room within the framework for expansion and elaboration to include other components of mathematics, such as decimal fractions.

The developmental framework shows how children's understanding of the number system becomes increasingly sophisticated as their thinking develops. The framework is designed to help teachers differentiate among their students on the basis of the children's understanding of the number system. Each stage in the framework includes three different components: the number concept itself, the spoken number word which refers to the concept orally, and the written numeral which records the concept in written form using symbols. Each of the three components is linked to the other two, although initially (in the years prior to school entry) knowledge of written numerals may not be connected either to number concepts or to spoken number words (*see* dotted lines in figure 2).

NUMBER CONCEPTS

Children need to build a rich network of ideas about the patterns and relationships among numbers. Central to this network of relationships is the number concept itself, which may be established in a variety of ways. For example, counting is an important process by which the answer to a "How many?" question can be determined. Rochel Gelman has identified several principles involved in the counting process, including the one-to-one principle (pairing off each object with a different number word while maintaining one-to-one correspondence between an object and a number word), the stable order principle (producing the sequence of number words in a consistent order each time), and the cardinality principle (the idea that the last number word produced during counting designates the total number of objects in the group).[5] Slightly more challenging than simply counting a given group of objects is making a group of objects on request (that is, forming sets). Research has shown that children's competence with forming sets when they begin school at 5 years of age is strongly related to their success in mathematics several years later.[6] Everyday life is full of opportunities for children to learn how to create small groups of objects, for instance getting a particular number of carrots or potatoes for dinner, setting the table, ensuring there are enough chairs for everyone to sit down, sharing out special food among friends.[7]

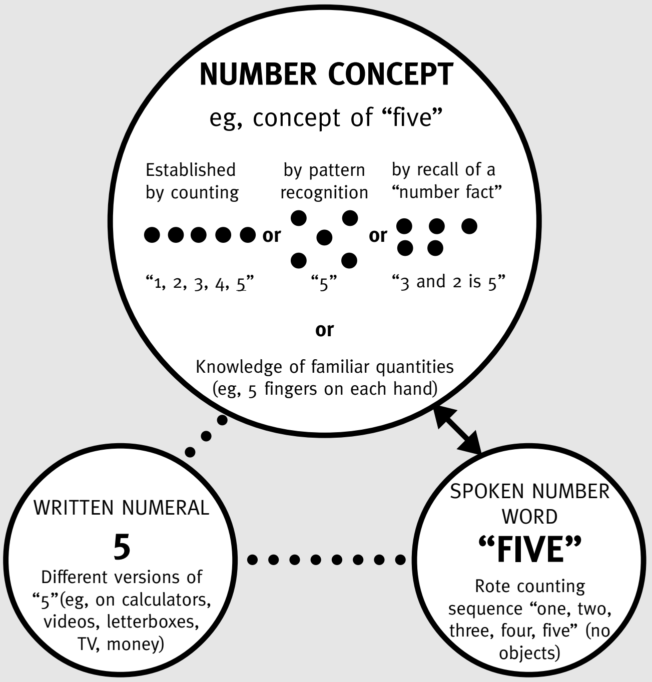
Another way of establishing quantity can be through recognition of a stylised number pattern, such as that depicted on dice or dominoes (that is, pattern recognition, sometimes called subitising). Such recognition is almost immediate and does not involve counting. Board games, card games, and dominoes also provide enjoyable contexts for learning about quantities.[8]

Figure 1: Developmental framework for the acquisition of numeracy

Construction of a strong *unitary* concept of numbers, then a shift to *multi-unit* understanding

1. **Unitary concept**
   (for single-digit and multi-digit numbers)
   Knowledge of number word sequences, counting processes, part-whole relationships, numerals and number patterns
2. **Ten-structured concept**
   Whole decade partitioned into units of 10 ones
3. **Multi-unit concept**
   Units of tens and ones counted separately, and can be traded and exchanged (eg, 10 ones for one 10, or one 10 for 10 ones)
4. **Extended multi-unit concept**
   Units can be any power of ten

Figure 2: A variety of ways of understanding the concept "five": a unitary concept



Another source of knowledge about quantities is the familiar quantities that children come to "just know" (for example, two hands, three wheels on a tricycle, four legs on a cat, five fingers on a hand). Most children learn very early the meaning of "two" without needing to count (wanting to have one biscuit for each hand is a good example). Gradually, with experience, they learn to recognise larger groups of objects, like "three", "four", and "five", without needing to count them.

Children also come to "just know" certain number combinations involving small numbers (that is, number facts). Children learn about combinations of small numbers first (for example, "two and one more makes three"). "Doubles" can be used to work out some of the other combinations. For example, "two and two makes four, and another one makes five" could eventually lead to "three and two makes five altogether" (that is, "derived number facts").

SPOKEN NUMBER WORDS

As number concepts develop, so too does the language which enables children to communicate their understanding of number concepts to others (spoken number words). Children learn the names of the number words which in their own culture are used to talk about and refer to quantities. Evidence from earlier research shows that sometimes the connections between number concepts and the spoken number words that refers to those concept may be slightly faulty to begin with. Some young children seem to go through a stage of being able to hold up the correct number of fingers corresponding to a stylised number pattern presented on a dot-dice, but say the wrong number name (for example, they see a dot-dice pattern of three and hold up three fingers, but say "four"). This idiosyncratic response pattern, which was marked among younger 4-year-olds, disappeared as the children established the correct connection between number concepts and the spoken number words used to refer to those concept.[9]

The conventional sequence of spoken number words, while meaningless on its own, is an essential tool in building an understanding of the number system.[10] Often children's errors in rote counting reflect their understanding of the rules by which the number system operates, as in the production of novel number words such as "twenty-ten", and "twenty-eleven", or consistently stopping at a number ending in nine or zero. There is a huge literature on the issue of language and culture and its impact on children's learning about the number system.[11] For example, Asian languages differ from languages like English and French in having a very systematic and transparent structure for their number word sequence (for example, "12" in Japanese is "juni" or "ten-two"). This may be one of the reasons that Asian children tend to achieve very highly in mathematics compared to children from Western countries.

WRITTEN NUMERALS

Written numerals are the symbols which record number concepts in written form. Children initially learn the names of numerals which are familiar to them (for example, the "1", "2", "3", and "4" which identify each of the television channels, the numbers on people's letter boxes, the numeral "5" on the birthday hats made for children just before they graduate from an early childhood centre to school on their fifth birthdays). Often numerals are used as labels which identify familiar objects and help to distinguish them from other objects (for example, telephone numbers, car number plates), rather than as a record of quantity.[12] For this reason, knowledge of numerals is initially quite separate and distinct from knowledge about specific quantities (*see* dotted lines connecting the numeral "5" to the concept of "five" and the spoken number word "five" in figure 2). Only gradually do children come to understand the connection between a written numeral and the quantity which it represents.

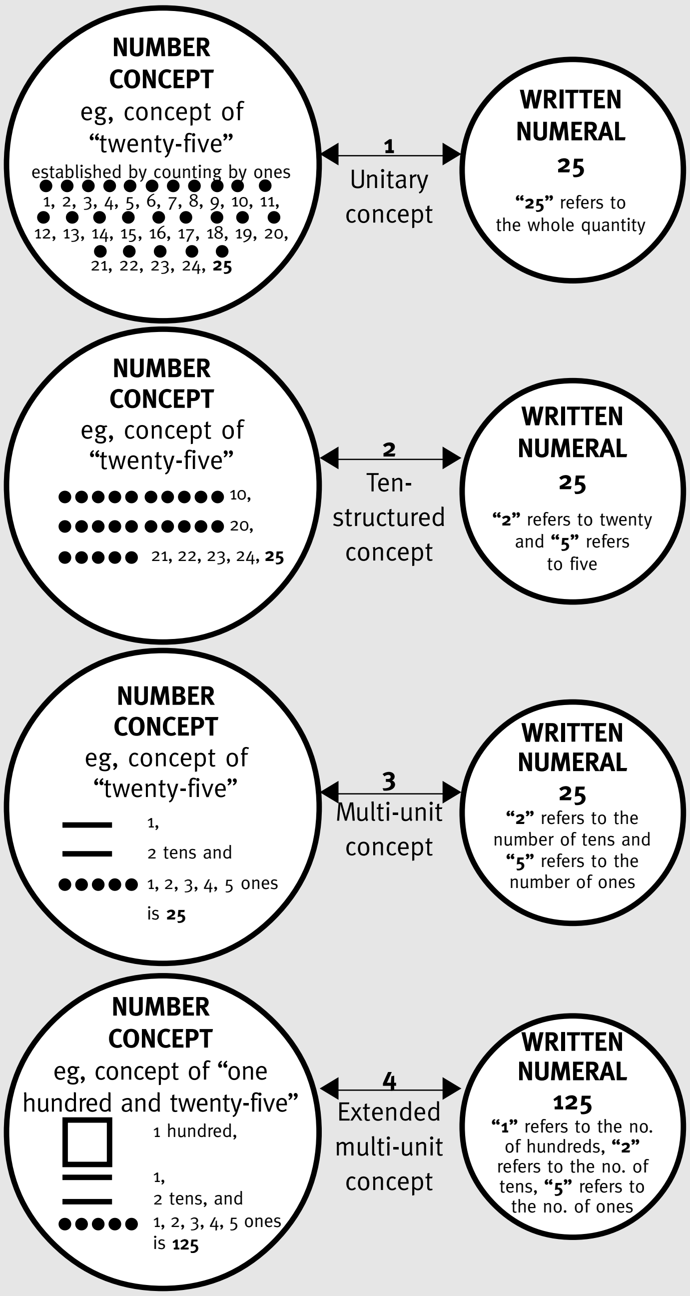
DEVELOPMENTAL PROGRESSION

The developmental framework begins with a unitary (by ones) concept of numbers, then moves through a transitional stage of ten-structured thinking, which leads to a multi-unit (by tens and ones) concept of numbers (*see* figure 1). The fourth stage in the framework is an extended multi-unit stage in which the idea of multiple units is extended to units larger than ten (for example, hundreds, thousands, and so on). Extended multi-unit understanding makes it possible for children to work with large numbers efficiently using advanced understanding of place-value.

UNITARY CONCEPT

Building a strong unitary concept of numbers is an essential first step in coming to understand the number system, particularly as it relates to small quantities (that is, those represented by single-digit numerals). The magnitude of a quantity can be ascertained by unitary counting. Children's counting gradually shifts from a dependence on physical materials to mental counting, and eventually to an understanding that a number can be taken to

Figure 3: Example of each stage in the developmental framework for numeracy acquisition



imply a count (without actually needing to count any objects),[13] thus enabling the use of "counting on" instead of "counting all" to solve problems involving addition.[14]

Part-whole relationships among numbers also become important at this stage. This refers to the idea that numbers are composed of parts which together make up the whole number (for example, 3 and 2 makes 5). Children gradually learn to recall "number facts" to solve addition and subtraction problems instead of using counting strategies. Many writers have stressed the importance for children of coming to understand the "additive composition of numbers", and recommend giving children lots of experiences with single-digit sums and differences to 18 to build a strong network of relationships among numbers which can form the base on which more advanced understanding builds.[15] Children need to understand that there are many different ways to construct a particular number (for example, 18 can be made from 9 and 9, 10 and 8, 11 and 7, and so on). Knowledge of the numbers pairs which make ten (for example, 7 and 3, 8 and 2) are thought to be particularly important also.[16]

Not all word problems have the same structure. Some researchers have explored the many different structures which addition and subtraction word problems can have.[17] For example, some problems involve "change" while others involve "combining" or "comparing" or "equalising". Change problems are often posed using the structure "result unknown" (for example, Jo has 6 lollies and Ann has 4 lollies, how many do they have altogether?). Alternative structures such as "start unknown" (Jo had some lollies then her mother gave her 4 lollies and now she has 10 lollies. How many lollies did she have to begin with?) or "change unknown" (Jo had 6 lollies then her mother gave her some more lollies, and now she has 10 lollies. How many lollies did her mother give her?) are more challenging for children than problems with a "result unknown" structure.

Initially children use a unitary concept of number for multi-digit numbers (*see* figure 3). With a unitary concept, the separate number words and digits have no meaning on their own, and the entire number word or numeral refers to the whole quantity. Some writers argue that children have a "mental number line", in which numbers in the sequence correspond to positions along a string, with individual positions linked by a "next" or a "one-more-than" relationship.[18]

There is some debate about whether or not understanding part-whole relationships constitutes a superior way of thinking about numbers than a mental number line model. Both part-whole and mental number line conceptions are necessary if children are to have a good understanding of the relationships among numbers, and be able to use this knowledge in flexible ways.

TEN-STRUCTURED CONCEPT

The key feature of the second stage of this developmental framework is the shift which must be made from a unitary way of thinking about numbers, towards a multi-unit conception. In this transitional stage of ten-structured thinking, part-whole understanding is refined to take into account the special significance of ten. Initially, a quantity can be partitioned into a whole decade plus extra ones (for example, 25 as 20 plus 5). The decade part of the quantity can then be partitioned into units of ten ones (for example, 10, 20). The only way to ascertain how many tens there are is to count by tens, keeping track of the number of counts. During this stage, numbers come to be seen as composites of ten and ones. Learning about the relationships between numbers and their nearest whole decades is particularly useful for solving addition or subtraction problems for example, 17 as 7 beyond 10 and 3 short of 20).

In contrast to the unitary stage, which can be characterised by a mental number line model with numbers running from left to right as they get larger (that is, one-dimensional and linear), the ten-structured stage can be thought of as two-dimensional (like a hundreds board), with numbers increasing by "ones" from left to right, and increasing by "tens" from the top downwards.[19] The digit in the tens column refers to the whole decade and the digit in the ones column refers to the leftover ones (for example, "2" in 25 means 20, and "5" means 5 ones).

MULTI-UNIT (TENS AND ONES) CONCEPT

In this stage, the units of tens and ones are counted separately. The principle of "trade and exchange", the idea that a number is not altered by legal exchanges (for example, 1 ten for 10 ones or 10 ones for 1 ten) can be used to solve problems where the ones in a quantity being subtracted exceed the ones in the quantity from which it is being subtracted and "renaming" is required. As numbers get larger, the advantages of having a multi-unit concept over a unitary concept become greater. Instead of counting the entire quantity in ones, units larger than one (for example, tens) can be counted as single units, then the leftover ones can be counted on at the end. Unlike the unitary and ten-structured concepts in earlier stages, multi-unit concepts include knowledge of the base name (for example, "25" as 2 tens and 5 ones). At this stage, there is a direct link between each digit in a multi-digit numeral and the quantity to which it refers (for example, the "2" in "25" means two tens, while the "5" means five ones).

The partitioning (or breaking down) of multi-digit numbers into a tens part and a ones part is sometimes referred to as "unique partitioning".[20] This contrasts with "multiple partitionings", which involve groupings other than the tens and ones which are immediately obvious (for example, unique partitioning of 47 is 4 tens and 7 ones, whereas multiple partitioning of 47 could include 3 tens and 17 ones, 11 fours and 3 ones, and so on). Understanding the equivalence of several partitionings is thought to be essential for understanding about "borrowing" which results in more than 9 of a particular unit being present, at least temporarily, without changing the total value of the quantity.[21] Children need to understand that exchanges (for example, 1 ten for 10 ones) that maintain equivalence do not affect the quantity. Children can be credited with a complete understanding of the possibilities for multiple representation only when they are no longer dependent on counting to establish that two identical collections of objects have the same number.

EXTENDED MULTI-UNIT (HUNDREDS, TENS, AND ONES) CONCEPT

In this stage, the idea that units can be different sizes and must be counted separately is generalised to larger units such as hundreds, thousands, and beyond. It takes time for the ideas encompassed in multi-unit concepts involving tens and ones to be generalised to larger units such as hundreds and thousands. This is evident in children's tendency to use groups of ten to form sets of 125 and 267, even though groups of one hundred were readily available and would have been more efficient than using ten groups of ten.[22]

As well as partitioning multi-digit numbers in various ways, children can be helped to see the relationships between a particular number and other numbers (for example, the number 265 is: 15 more than 250, 65 more than 200, and 35 less than 300), to compare magnitudes (for example, 265 is large compared to 13, about the same as 273, small compared with 894), and to make connections with real-life (practical) experiences (for example, the number of children in the hall or the cost of a television set in dollars).[23]

LEVELS OF ABSTRACTION

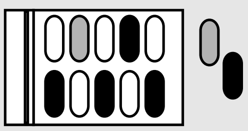
It is sometimes assumed that all ways of grouping of materials into tens (and larger units) are equivalent. Different kinds of grouping are characterised by different degrees of abstraction/concreteness, and these can be ordered by level (*see* figure 4).[24] At the lowest level of abstraction (that is, the most concrete level), 10 ones can be grouped into a ten (for example, put into a zip-lock bag or bound together with a rubber band). These groupings can be easily reversed and the objects "unpacked". The second level of abstraction involves trading in 10 ones for a "pregrouped" ten (for example, a "long" for 10 small cubes, or a pregrouped 10 marker). The third level involves trading in 10 ones for a different-looking 10 marker (for example, a $10 note for 10 $1 coins, or a different-coloured counter for 10 counters of a particular colour). The fourth (highest) level involves trading in 10 ones for an identical marker that represents ten simply by virtue of its position. It has been argued that using a different-looking 10 marker (third level) may help low-ability students especially to bridge the gap between highly concrete size embodiments typical at the lowest level of abstraction, and the positional feature characteristic of the highest level of abstraction. It is the significance of position which needs to be recognised by children if they are to understand place value; that is, the meaning of an individual digit by virtue of its position in a multi-digit numeral. For example, the digit "2" has different meanings in 62, 24, and 2005 (that is, two, twenty, and two thousand, respectively). Children need to understand place value if they are to use the number system accurately and efficiently to solve problems involving large numbers.

Figure 4: Increasingly abstract models for "twelve" as identified by Baroody[27]

TEN ONES CAN BE:

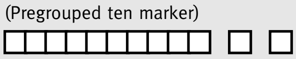
1. **Grouped into ten**

   (unpackable grouping of ten)



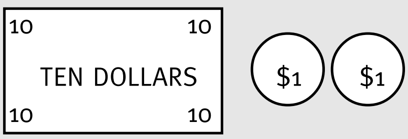
2. **Traded for a pregrouped ten**

   (Pregrouped ten marker)



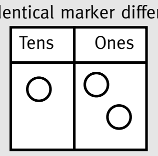
3. **Traded for a different-looking ten marker**

   (Different-looking ten marker)



4. **Traded for an identical marker that represents ten by virtue of its position**

   (Identical marker differing only in position)



Symbolic representation

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PROPERTIES OF THE NUMBER SYSTEM

The positional property of the number system is just one of several properties which must be understood to understand the meaning of a large multi-digit number.[25] The four properties of the number system are outlined below:

- *Positional property*. The quantities represented by the individual digits are determined by the position they hold in the whole numeral.
- *Base-ten property.* The values of a position increases in powers of 10 from right to left.
- *Multiplicative property.* The value of an individual digit is found by multiplying the face value of the digit by the value assigned to its position.
- *Additive property.* The quantity represented by the whole numeral is the sum of the values represented by the individual digits.

RESEARCH EVIDENCE TO SUPPORT THE FRAMEWORK

Recent research with a representative group of 97 nine-year-olds has provided important evidence to support the framework outlined here.[26] A group of 19 children were identified as having unitary understanding of the number system. Most of these children were able to read and write two- and three-digit numerals, count by tens to at least 100, give the number "one more than" a one- or two-digit number, and do addition problems with combinations making 10, either by recalling a number fact or by using a counting strategy. Most could also add a single-digit number to 10, providing they could use a counting strategy. The 20 children identified as having ten-structured understanding were able to do all or most of the tasks which children with unitary understanding could do, and as well they could use groupings of 10 (that is, clear plastic bags each containing 10 objects and $10 notes) to create sets of 31 and 125 more quickly and accurately. The 19 children identified as having multi-unit understanding were able to do all or most of the tasks already described, and as well showed evidence of understanding place value. For example, most could show the link between individual digits in a multi-digit numeral and the objects to which the digits referred, and give the number which is 10 more than a one- or two-digit number. Thirty-nine children were identified as having extended multi-unit understanding. These children were very proficient at all other tasks, but as well could give the numbers which are one hundred, one thousand, and ten thousand more than the one- to five-digit numbers specified. It was the children in this fourth group who had the best grasp of place-value understanding, and this was reflected in their performance on a variety of different tasks. They were also the best at performing operations (particularly addition) presented in a variety of contexts, including verbal money problems, problems involving place value blocks, written computation (presented vertically), and a novel task in which problems were presented diagrammatically using the positional feature of the number system (that is, tens-ones and hundreds-tens-ones boxes, *see* 4. in figure 4.)

HELPING CHILDREN REACH MULTI-UNIT UNDERSTANDING

Several writers have emphasised the importance of building multi-unit understanding using concrete materials as soon as children begin using two-digit numbers.[27] It has been suggested that prolonged practice with unitary concepts (a likely scenario for low-ability students) may interfere with learning multi-unit concepts.[28] The "teen" numbers in English can be confusing (because of the lack of correspondence between the written numeral and the way the number is said: "fourteen" is written as 14 with the "teen" to the left of the "four", whereas "forty-one" is written 41), and some of the decade names don't correspond exactly to the units they stand for (for example, "twen-ty" for two tens and "thir-ty" for three tens don't have the regular pattern shown by "six-ty" for six tens).[29]

Having a variety of place-value embodiments seems to be a key factor in children's learning about place value. One study found that the 6- and 7-year-olds who made the best progress towards understanding place value were in classrooms in which they were given a choice of activities (within limits set by the teacher) based on real-life experiences.[30] They also had access to a variety of place value embodiments and models, and were given many opportunities to share and discuss their mathematical ideas with both the teacher and their peers. Children who made the least progress in place value understanding were given few place-value embodiments (for example, only the hundreds board to explore patterns, and counters for counting and grouping in tens), and few, if any, opportunities to share their findings or their methods with the teacher or their peers.

SUGGESTIONS FOR TEACHERS

The literature on place value has shown the importance of understanding the additive composition of the number system, and the value of concrete materials, particularly those which can be grouped and ungrouped as needed. Children need to be helped to appreciate the additive composition of numbers, the special significance of the number "10" for our base-ten number system, and to understand the way in which large numbers are composed of multiples of 10. It is important for children to have lots of experience with materials of different kinds. There are many different materials that can help children understand the ten-structured nature of the number system and appreciate the benefits of using multiple units for working with large numbers.

PLAY MONEY

Money provides a familiar context in which children can learn about working with units of different sizes.[31] Money from the *Reserve Bank of Toyland* includes $1 coins, as well as $10 and $100 notes. The money in the board game *Monopoly* includes $1, $10, and $100 notes also. Children quickly come to see how a $10 note is much quicker to get than 10 $1 coins or notes. Money has the added advantage that the numeral which corresponds to the quantity is printed on the note. This may be the reason that children more readily use multiple units of money for large quantities than objects grouped by tens and hundreds in plastic bags and boxes.[32]

GROUPED OBJECTS IN PLASTIC BAGS

Small objects which can be grouped in tens and placed in clear zip-lock bags are very useful (10 plastic beans fit very neatly into the 50 mm by 75 mm bags which are available from stationery shops). The advantage of using clear plastic bags for grouping the objects is that the ten-ness is immediately transparent (literally) and the ten objects inside the bag can be counted to check their number without actually opening the bag. In this way, zip-lock bags have an advantage over opaque containers (for example, film canisters) for grouping small objects in tens because the objects can be more easily counted while still in the bags. Also, because the zip-lock bags can be opened, a bag of 10 objects can be unpacked into 10 units of one, using the principle of trade and exchange. Larger units can be created by stacking 10 bags of 10 objects in a larger container with clear sides (for example, a 400 ml box with a lid) to make one hundred, and putting ten 400 ml boxes into a larger plastic container to make one thousand.

PARTITIONING BOXES

Partitioning boxes provide a good way of showing alternative ways of breaking down numbers into their component parts (*see* figure 5).[33] The numeral is written in the small box in the centre and the corresponding number of objects is partitioned between the two sections of the box. Multiple partitioning boxes enable children to show a variety ways of making a particular number. Ungrouped objects may be used initially, then grouped objects introduced at a later stage to make it easier dealing with large numbers.

TENS FRAMES

Several writers advocate the use of tens frames for helping children to appreciate part-whole relationships (*see* figure 5).[34] The idea of tens frames can be extended to encompass a double-tens frame, to help children learn about the relationships between numbers up to 20, and a ten-by-tens frame, which helps children learn about part-whole relationships with numbers up to 100. Images of tens frames can be made into overhead transparencies and objects can be used to create shadows within the compartments. Alternatively, 50 mm by 75 mm bags containing 10 beans can be lain over blocks of 10 on the ten-by-ten frame, providing the most concrete embodiment of ten. At a more abstract level, pregrouped ten-ones cards or one-ten cards, can be lain over the blocks of 10 on the ten-by-ten frame. Being able to put down blocks of 10 on ten-by-ten frame enables children to construct larger numbers more quickly and easily than if they counted out the objects by ones.

PLACE-VALUE MATS

Another tool to help children to appreciate the base-ten structure of the number system is the place-value mat.[35] The right, unshaded side of the place-value mat is the ones side, and the left shaded side is the tens side (*see* figure 6). Place-value mats can be used to model numbers at the most abstract level, with a larger unit (for example, ten) distinguished only by its position (that is, the left side of the place-value mat). Place-value mats can be made for tens and ones only, or for hundreds, tens, and ones, or larger denominations.

SLAVONIC ABACUS

Another piece of equipment for helping children appreciate the ten-structured nature of the number system is the Slavonic abacus, which consists of 10 parallel rows of 10 beads, with a change of colour after each five (*see* figure 6).[36] A quantity is represented on the abacus by pushing the requisite number of beads from one side of the abacus to the other. For example, "28" would be shown by having the top two rows of 10 beads and eight of the next row all pushed to the left. Looking at the right side of the abacus shows how many beads are needed to make the quantity up to the next whole decade (that is, two beads will make 30) or to the entire century (that is, the seven rows of 10 beads plus the two single beads on the right side). The change of colour after five enables children to subitise quantities of five or less and learn the number combinations which make 10, using five as a benchmark. For example, the eight beads in the example above can be quickly recognised as made up of 5 and 3.

QUANTITY PICTURES

A paper abacus can be easily adapted from the Slavonic abacus to create a permanent

Figure 5: Examples of a partitioning box (top), a tens frame (middle), and a ten-by-tens frame (bottom)

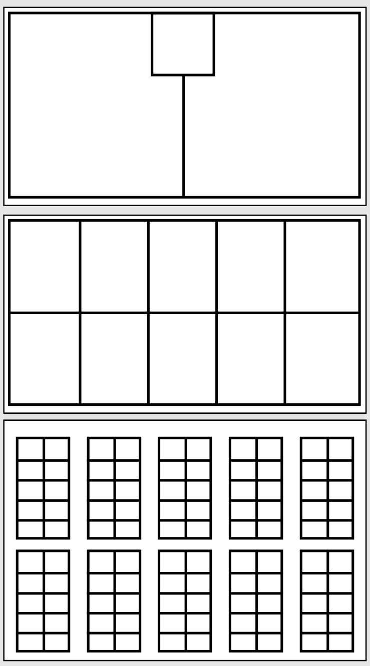
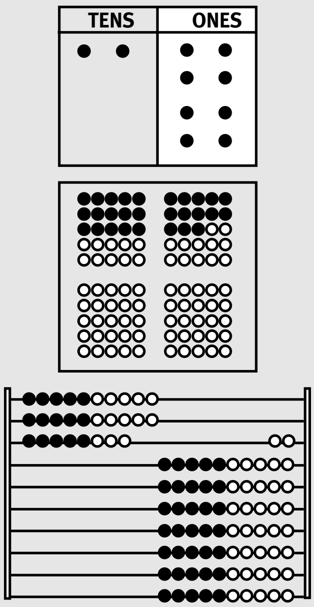


Figure 6: Examples of a place value mat (top), a quantity picture (middle) and a Slavonic abacus (bottom)
The Slavonic abacus and quantity pictures are described in Grauberg, see note 36



record of a quantity (*see* figure 6). Using a slight break between the rows and columns of five as well as a colour change enables children to subitise quantities of five and less, and learn the number combinations which make 10, using five as a benchmark. Children can show a quantity by colouring in the circles or by drawing a boundary around the requisite number of circles. Quantity pictures can be made using other shapes such as boxes. A variety of worksheets for quantity pictures can be created which allow quantities of different magnitudes to be shown (*see* figure 7).[33] Once children are adept at using a ten-by-ten grid for quantity pictures, the grouping of five within 10 is probably not necessary. It is then possible to create quantity pictures for numbers in the hundreds and thousands on worksheets which have multiple ten-by-ten grids on them.

TEN-STRUCTURED BEAD STRINGS

All of the materials described so far are useful for building and strengthening children's part-whole understanding of the number system. Maintaining the links between part-whole relationships and the number line are also important. Researchers in the Netherlands have found that using ten-structured bead strings to model the empty number line was very effective.[37] The bead strings are made up of 100 beads, arranged in strings of 10, alternating in colour. The beads can be any size, but 10 mm beads threaded onto nylon cord produce strings of manageable lengths, and at a reasonable cost. Because of the flexibility of the cord, multiple strings (for example, 10 strings to make one thousand) can be manipulated easily by children on a table-top and bent for easy storage in a container. Children with limited understanding of the number system can use 20-bead strings in which the beads are arranged in fives in alternating colours (*see* figure 8). These strings with alternating fives have the same kinds of advantages as the colour change used in the Slavonic abacus. They enable children to subitise up to five and learn the number combinations which make 10.

TEN-STICKS AND DOTS

A simple, economical system for recording quantities as tens and ones was introduced to 6-year-old children by Karen Fuson and her colleagues (*see* figure 9).[38] The system was designed to help children see objects grouped in tens, and relate these ten-groupings and the leftover ones to number words and written numerals. The system used ten-sticks to represent tens, and dots to represent ones (boxes could be used to represent hundreds). Initially children made dots in columns of 10 to make a record of the objects in various collections. They counted by ones as they made these columns of 10 dots. When they had fewer than 10 dots left, they made a horizontal row of dots (often with a space between the first five dots and the last four dots to make it easier to see how many dots there were). To check a quantity, children could then count all the dots by ones (unitary concept), count the columns *by* tens (ten-structured concept), or count the columns *of* tens (multi-unit concept). When many children could make these drawings confidently, the 10 dots in a column were connected by a line drawn through them as the counting *by* tens or *of* tens was done. Eventually only the vertical stick was drawn to show a 10. For addition problems, dots were combined to make another ten-stick, shown by an elliptical line around the 10 dots. For subtraction problems which involved regrouping, a ten-stick was opened using an arrow pointing to an ellipse containing 10 dots. The researchers found when they assessed the children who had used this system of recording that their errors mostly involved miscounting (a procedural problem) rather than misconceptions or misunderstanding.

HOW SHOULD PLACE VALUE BE TAUGHT?

Some writers argue that learning procedures for adding and subtracting multi-digit numbers can provide opportunities for motivating and supporting the development of base-ten number concepts.[39] Invented strategies are thought to provide a useful context for advancing children's base-ten understanding, because grouping in tens is made so explicit in invented strategies. In the process of inventing their own strategies for solving problems, children come to recognise the special significance of ten and understand that numbers are composites of tens and ones. When left to their own devices, many children invent left-to-right strategies for solving multi-digit addition and subtraction problem. The value of encouraging children to invent their own strategies rather than teaching them the standard algorithms has been demonstrated in a recent study which found that students who began by using invented strategies demonstrated better knowledge of base-ten number concepts and were more successful in extending their knowledge to new situations, compared with students who learned standard algorithms initially.[40]

WHEN SHOULD PLACE VALUE BE TAUGHT?

Research on place-value understanding provides a rough guide to how many children in a particular age group we might expect to understand place value. For example, in one study just over a quarter (27 percent) of 8-year-olds and under half (44 percent) of 9-year-olds had a good understanding of place value. Some writers are extremely cautious about the age at which place value concepts should be introduced to children. For example, Constance

Figure 7: Examples of worksheets for quantity pictures for quantities less than 100 (top) and between 100 and 400 (bottom)

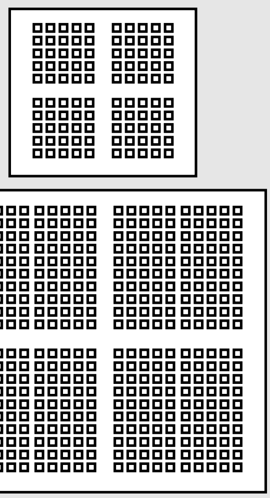
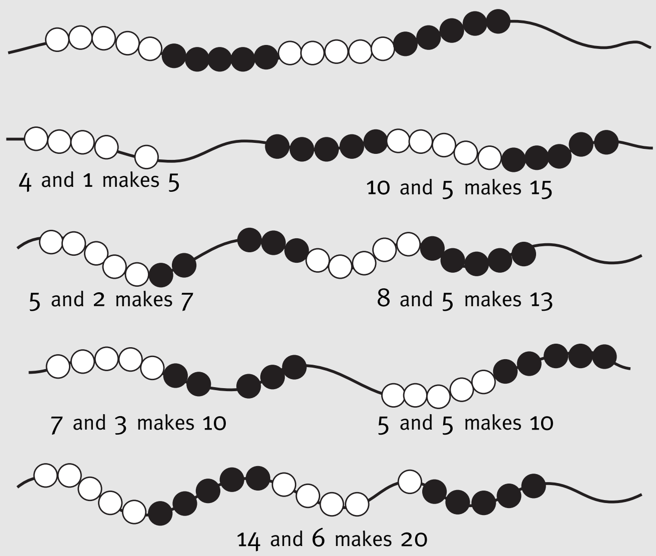


Figure 8: Using a 20-bead beadstring

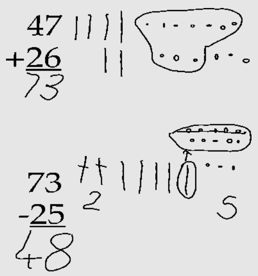


Kamii argues that place-value instruction should be delayed until children have constructed the relationships between numbers in the number-word sequence, (by repetition of the +1 operation) and can partition wholes in many different ways (part-whole relationships).[41] Kamii argues strongly against activities which attempt to teach the conventional system of writing numbers from *outside* the child (for example, bundling of ten-sticks using rubber bands), on the grounds that a child needs to construct the system of tens on the system of ones, *within* him or herself by "introducing a mental relationship among the objects to quantify them numerically". This contrasts with other theorists who see the number system as a culturally derived tool which depends on parents, teachers, books, etc (that is, social transmission) to help children learn about it.[42]

Other writers have talked about the need for children to be given opportunities to construct a deep understanding of base-ten place value concepts, a rich repertoire of flexible, mental calculation strategies, and a set of adequate beliefs about arithmetic problem-solving.[43] They argue that the introduction of the written algorithms should be delayed until the third or even the fourth year at school.

There is reasonable unanimity among researchers about the idea that children should be given lots of experiences with partitioning and recombining numbers in order to build up a network of number relationships, on which an understanding of the special significance of ten can be based. Teachers need to recognise the importance for place-value understanding of this work with "ten-ness", and ensure that their instructional methods are sensitive to children's existing knowledge. It is vital that a teacher understands what developmental stage a child is at in terms of multi-digit number understanding, and uses that information to plan activities and tasks which are appropriate for that child. For example, if children are not secure in their understanding of the numbers represented by single-digit numerals, then it makes no sense to give them multi-digit numbers. If children are still grappling with the idea that numbers can be partitioned into several parts, then it is too soon to try and help them understand the special significance of ten and the partitioning of numbers into a tens part and a ones part. There is no point in focusing on the idea of trade-and-exchange, if the child does not yet understand that numbers are composites of tens and ones. Children cannot begin working with place-value understanding for numbers in the hundreds and thousands if they do not yet understand the idea that tens are a different kind of unit from ones. Children are likely to make the best progress if their teachers can find out how they are currently thinking, then challenge and support them in moving towards more sophisticated ways of thinking and working.

Figure 9: An example of addition and subtraction using ten-sticks and dots



FITTING THE FRAMEWORK WITH THE MATHEMATICS CURRICULUM

Because the framework focuses on the number system, it may appear to be relevant only to the number strand of the curriculum. However, there is potential for each of the curriculum strands to enhance or be enhanced by children's understanding of the number system: the number system provides a means for expressing mathematical ideas and interpreting written presentations of mathematics (*mathematical processes*); it is vital for students in developing an understanding of numbers, the ways they are represented, and the quantities for which they stand, as well as accuracy, efficiency, and confidence in calculating (*number*); it is necessary for understanding and using systems of measurement (particularly metric) (*measurement*); its understanding can be greatly enhanced by the use of geometric models which enable students to visualise quantities arranged in various ways (*geometry*); it encompasses many patterns and relationships which can be represented and communicated using symbols and diagrams (*algebra*); and it is a vital tool for use in organising, analysing, and presenting data (*statistics*).

SOME RECOMMENDATIONS

- that children be given lots of experiences at constructing numbers which foster their understanding of part-whole relationships, particularly single-digit sums and differences to 18, using physical materials and problems requiring mental problem solving;
- that children are supported in constructing number concepts based on relationships with ten, or multiples of ten;
- that children are helped to construct place-value understanding as a particular instance of part-whole relationships involving groupings of tens and ones;
- that children be encouraged to develop their own strategies for adding and subtracting multi-digit numbers;
- that children are helped to understand the principle of "trade and exchange", the idea that a unit of one denomination can be exchanged for 10 units of another denomination.

NOTES

JENNY YOUNG-LOVERIDGE is a senior lecturer in the Education Studies Department at the University of Waikato and a member of the mathematics subgroup of the Mathematics and Science Taskforce. She has been doing research on the development of children's mathematical thinking for almost 15 years. Contact: Education Studies Department, University of Waikato, Hamilton, New Zealand. Phone: +64 7 838 4353. Fax: +64 7 838 4434. E-mail: educ2233@waikato.ac.nz

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1. See page 7 of:

Ministry of Education. (1992). *Mathematics in the New Zealand curriculum*. Wellington, NZ: Learning Media.

2. Children's understanding about numbers:

Bergeron, J. C., & Herscovics, N. (1990). Psychological aspects of learning early arithmetic. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and cognition: A research synthesis by the International Group for the Psychology of Mathematics Education*. Cambridge: Cambridge University Press.

Boulton-Lewis, G. (1996). Representations of place value knowledge and implications for teaching addition and subtraction. In J. Mulligan & M. Mitchelmore (Eds.), *Children's number learning: A research monograph of MERGA/AAMT* (pp. 75-88). Adelaide, SA: Australian Association of Mathematics Teachers.

Fuson, K. C., Smith, S. T., & Cicero, A. M. L. (1997a). Supporting Latino first graders' ten-structured thinking in urban classrooms. *Journal for Research in Mathematics Education, 28* (6), 738–766.

Fuson, K. C., Wearne, D., Hiebert, J. C., Murray, H. G., Human, P. G., Olivier, A. I., Carpenter, T. P., & Fennema, E. (1997b). Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction. *Journal for Research in Mathematics Education, 28* (2), 130–162.

Hiebert, J. (1988). A theory of developing competence with written mathematical symbols. *Educational Studies in Mathematics, 19* (3), 333–356.

Jones, G. A., Thorton, C. A., Putt, I. J., Hill, K. M., Mogill, A. T., Rich, B. S., & Van Zoest, L. R. (1996). Multi-digit number sense: A framework for instruction and assessment. *Journal for Research in Mathematics Education, 27* (3), 310–336.

Kamii, C. K. (1985). *Young children reinvent arithmetic: Implications of Piaget's theory*. New York: Teachers College Press.

Kamii, C. K. (1989). *Young children continue to reinvent arithmetic—2nd Grade: Implications of Piaget's theory*. New York: Teachers College Press.

Resnick, L. (1983). A developmental theory of number understanding. In H. P. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 109–151). New York: Academic Press.

Ross, S. H. (1989). Parts, wholes, and place value: A developmental view. *Arithmetic Teacher, 36* (6), 47–51.

3. Cognitively guided instruction (CGI):

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: A longitudinal study of invention and understanding in children's multi-digit addition and subtraction. *Elementary School Journal, 97* (1), 3–20.

4. An Australian project, Count Me In Too, seems to have had a similar effect, see:

Wright, R. (1997). *Mathematics recovery leaders' handbook*. Lismore, NSW: Southern Cross University.

5. The principles of counting:

Gelman, R., & Gallistel, C. R. (1978). *The child's understanding of number*. Cambridge, MA: Harvard Press.

6. Competence with forming sets and later success:

Young-Loveridge, J. M. (1991). *The development of children's number concepts from ages five to nine: Volume I: Report of findings*. Hamilton: University of Waikato.

7. Everyday experiences and the number system:

Young-Loveridge, J. M. (1989). The relationship between children's home experiences and their mathematical skills on entry to school. *Early Child Development and Care, 43*, 43–59.

8. Number books and copy masters for games:

Young-Loveridge, J., & Peters, S. (1994). *A handbook of number books and games: From the EMI-5s Study*. Hamilton: University of Waikato.

9. Connecting number concepts with spoken names:

Young-Loveridge, J., & Peters, S. (1994, December). *Show and tell: Four-year-olds' understanding of number names and cardinal values*. Paper presented at the annual conference of the New Zealand Association for Research in Education, Christchurch.

10. Research on children's rote counting:

Fuson, K. C., Richards, J., & Briars, D. J. (1982). The acquisition and elaboration of the number word sequence. In C. J. Brainerd (Ed.), *Children's logical and mathematical cognition: Progress in cognitive developmental research*. New York: Springer-Verlag.

11. The impact of language and culture:

Young-Loveridge, J. M. (1998). *The development of place value understanding: A paper presented at the Mathematics Education Research Seminar*. Wellington: Ministry of Education.

12. The use of numerals to indicate cardinality, ordinality, and nominality:

Sinclair, H., & Sinclair, A. (1986). Children's master of written numerals and the construction of basic number concepts. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 59–74). Hillsdale, N J: Erlbaum.

13. This shift is sometimes called perceptual, figural, and abstract counting, see:

Steffe, L. P., & Cobb, P. (1988*). Construction of arithmetical meanings and strategies*. New York: Springer-Verlag.

14. From counting all, to counting on, to recall:

Carpenter, T. P., & Moser, J. M. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education, 15*, 179–202.

A similar model for multiplication and division:

Mulligan, J. T., & Mitchelmore, M. C. (1997). Young children's intuitive models of multiplication and division. *Journal for Research in Mathematics Education, 28* (3), 309–330.

15. Understanding the additive composition of numbers:

Fuson et al. (1997, a, b), and Resnick (1983), see note 2 above.

16. Knowledge of the number pairs which make ten is emphasised in Asian cultures; for a discussion, see:

Young-Loveridge (1998), see note 8 above.

17. Different types and structures of word problems:

Peterson, P. L., Fennema, E., & Carpenter, T. (1988/89). Using knowledge of how students think about mathematics. *Educational Leadership, 46* (4), 42–46.

Riley, M. S., Greeno, J. G., & Heller, J. I. (1983). Development of children's problem-solving ability in arithmetic. In H. P. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 153–196). New York: Academic Press.

18. That children have a mental number line:

Klein, A. S., Beishuizen, M., & Treffers, A. (1998). The empty number line in Dutch second grades: *Realistic* versus *Gradual* program design. *Journal for Research in Mathematics education, 29* (4), 443–464.

Resnick (1983), see note 2 above.

19. Viewing ten-structured thinking as two-dimensional:

Resnick (1983), see note 2 above.

20. Unique and multiple partitioning:

Resnick (1983), see note 2 above.

21. The value of partitioning and recombining numbers:

Fischer, F. E. (1990). A part-part-whole curriculum for teaching number in the kindergarten. *Journal for Research in Mathematics Education, 21* (3), 207–215.

Kamii (1989), see note 2 above.

Resnick (1983), see note 2 above.

22. Children's tendency to use groups of ten:

Young-Loveridge, J. M. (1998, December). *The development of place value understanding*. Paper presented at the annual conference of the New Zealand Association for Research in Education, Dunedin.

23. Beyond partitioning—relationships between numbers:

Thompson, C. (1990). Place value and larger numbers. In J. N. Payne (Ed.), *Mathematics for the young child* (pp. 89–108). Reston, VA: National Council of Teachers of Mathematics.

24. Levels of abstraction for concrete objects:

Baroody, A. J. (1990). How and when should place-value concepts be taught? *Journal for Research in Mathematics Education, 21* (4), 281–286.

25. Properties of the number system:

Ross (1989), see note 2 above.

26. Evidence to support the framework:

Young-Loveridge, J. (in press). A framework for the acquisition of numeracy. *Mathematics Education Research Journal.*

27. Advocating the use of concrete materials:

Faire, M. F. (1992). *Young children developing place value understanding and multi-digit number knowledge.* Unpublished masters thesis, University of Waikato.

Fuson, K. C. (1992). Research on whole number addition and subtraction. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 243–275). New York: Macmillan.

Verschaffel, L., & De Corte, E. (1996). Number and Arithmetic. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 99–137). Dordrecht: Kluwer Academic Publishers.

28. Potential damage of prolonged practice:

Baroody (1990), see note 24 above.

29. The challenges of the English number-word system:

Young-Loveridge (1998), see note 22 above.

30. The value of choice of activities:

Faire (1992), see note 27 above.

31. Using money to introduce place-value concepts:

Allardice, B. S., & Ginsburg, H. P. (1983). Children' psychological difficulties in mathematics. In H. P. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 319–350). New York: Academic Press.

32. Using groupings of ten rather than hundreds:

Young-Loveridge (1998), see note 22 above.

33. Full-sized copy masters available in:

Young-Loveridge, J. M. (1999). *MathMaker handbook: Resources to support the acquisition of numeracy in the primary years*. Hamilton: University of Waikato.

34. Advocates for the use of tens frames:

Baroody (1990), see note 24 above.

Payne, J. N., & Huinker, D. M. (1993). Early number and numeration. In R. J. Jensen (Ed.), *Research ideas for the classroom: Early childhood mathematics* (pp. 43–71). New York: Macmillan.

Van de Walle, J. (1990). Concepts of number. In J. N. Payne (Ed.), *Mathematics for the young child* (pp. 63–87). Reston, VA: National Council of Teachers of Mathematics

35. Place-value mats:

Bove, S. P. (1995). Place value: A vertical perspective. *Teaching Children Mathematics, 1* (9), 542–546.

36. Slavonic abacus and quantity pictures:

Grauberg, E. (1998). *Elementary mathematics and language difficulties: A book for teachers, therapists and parents*. London: Whurr Publishers.

37. Ten-structured bead strings:

Klein et al. (1998), see note 18 above.

38. The system of recording using ten-sticks and dots:

Fuson et al. (1997a), see note 2 above.

39. The value of invented strategies:

Carpenter, T. P., Franke, M. L., Jacobs, V. R., Fennema, E., & Empson, S. B. (1997). A longitudinal study of invention and understanding in children's multi-digit addition and subtraction. *Journal for Research in Mathematics Education, 29* (1), 3–20.

Chambers, D. L. (1996). Direct modelling and invented procedures: Building on students' informal strategies. *Teaching Children Mathematics, 3* (2), 92–95.

Kamii (1989), see note 2 above.

Rathmell, E. C. (1994). Planning for instruction involves focusing on children's thinking. *Arithmetic Teacher, 41* (6), 290–291.

40. Invented strategies versus standard algorithms:

Carpenter et al. (1997), see note 39 above.

41. Age groups and the understanding of place value:

Kamii (1985), see note 2 above.

42. That the number system is a culturally derived tool:

Ginsburg (1983), see note 31 above, identifies three systems of knowledge: 1, informal and natural; 2, informal and cultural; and 3, formal and cultural.

43. That children be given an opportunity to construct a deep understanding of base-ten place value concepts:

Verschaffel & De Corte (1996), see note 27 above.

from the acquisition of the mother. in both oral and written form (listening, tongue, which is intrinsically linked.Â Learning to learn skills require firstly the acquisition of the fundamental basic skills such as literacy, numeracy and ICT skills that are necessary for further learning. Building on these skills, an individual should be able to access, gain, process and assimilate new knowledge and skills. This requires effective management of oneâ€™s learning, career and work patterns, and, in particular, the ability to perse-vere with learning, to concentrate for extended periods and to reflect critically on the purposes and aims of learning. One achievement in numeracy is the acquisition of fact fluency. Fact fluency refers to the knowledge necessary to produce sums and differences in a flexible, timely and accurate manner. In the toddler years, children progressively acquire the requirements for fact fluency, often beginning with intuitive numbers (ex. know the meaning of one, two, three), leading to the ability to recognize that, for example, any set of three elements has a larger count than a set of two elements.Â Preschool children who have acquired the ability to count, name numbers, and make distinctions between different quantities tend to perform well on numerical tasks in kindergarten. Chapter 4 - Numeracy. Definition of the domain. Table 4.1 Numerate behaviour â€" key facets and their components.Â The scale of low proficiency literacy and numeracy in the adult population remains an issue for policy makers, particularly given the evolution of the labour market and the growing penetration of ICT in all areas of life. PIAAC is considerably expanding the information available regarding persons with low levels of literacy.