

# S4A5. Grad. Seminar on Representation theory Real Lie algebras

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Saturday 29 May 2010 and 12 June 2010,  
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## Schedule

	Date	Theme	Speaker
1	29.05.2010	Section 1.1	Nicolas
2	29.05.2010	Section 1.2	Nicolas
3	29.05.2010	Section 1.3	Nadja
4	29.05.2010	Section 1.4	Andreas
5	29.05.2010	Section 1.5	Alexander
6	29.05.2010	Section 1.6	Benjamin
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7	12.06.2010	Section 2.1	Stephan
8	12.06.2010	Section 2.2	Valentin
9	12.06.2010	Section 2.3	Sean
10	12.06.2010	Section 2.4	Michael
11	12.06.2010	Section 2.5	Nicolas
12	12.06.2010	Section 2.6	Nicolas

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We shall mainly use [Hel01] in which there is much more than what we need. In that book the classification of symmetric spaces is realised through the classification of real Lie algebras. We will try in the seminar to extract this last classification from the rest of the book.

# 1 First session: May 29 2010

## 1.1 Recollection on semisimple complex Lie algebras

**Date:** 29.05.2010.

**Timing:** 60 min.

**Reference:** Chapter III Sections 1-5 in Helgasson [Hel01] or [Bou71] and [Ser66] or last semester lectures, see <http://www.hausdorff-center.uni-bonn.de/people/perrin/perrin-v4a5>.

This talk will be a brief review on Lie algebras and especially the classification of semisimple complex Lie algebras. The Theorem of Chevalley giving a presentation over  $\mathbb{Z}$  of semisimple complex Lie algebras should be stated.

## 1.2 Few fact on complex Lie groups and their Lie algebras

**Date:** 29.05.2010.

**Timing:** 60 min.

**Reference:** Chapter II Sections 1-4 in Helgasson [Hel01].

In this talk, we should define the notion of complex Lie groups and see the relationship between, Lie groups and Lie algebras in particular through the exponential mapping. The existence of Haar measures for compact Lie group should be stated.

Most of the statements will be given without proof since these results do use techniques out of the scope of the seminar.

## 1.3 Adjoint groups and semisimple groups

**Date:** 29.05.2010.

**Timing:** 60 min.

**Reference:** Chapter II Sections 5-6 in Helgasson [Hel01].

The main topic in this talk is to define the adjoint group associated to a Lie algebra. In particular we shall be interested in the semisimple case where things work better. In section 5, do not deal with Proposition 5.4. In section 6, Proposition 6.1, Corollary 6.2, Corollary 6.3 and Proposition 6.4 where proved in last semester lectures. Therefore we can state them without proof. Theorem 6.9 shall be stated but not proved.

## 1.4 Real Lie algebras, complexification and real forms

**Date:** 29.05.2010.

**Timing:** 60 min.

**Reference:** Chapter III Section 6 of Helgasson [Hel01].

This talk will give the basic relationships between real Lie algebras and their complexification. We shall see that every semisimple complex Lie algebra has a real compact form. Example would probably be welcome here. For example, the case of  $\mathfrak{sl}_2(\mathbb{C})$  and its real Lie subalgebras  $\mathfrak{sl}_2(\mathbb{R})$  and  $\mathfrak{su}_2(\mathbb{R})$  should be explained.

## 1.5 Cartan decompositions

**Date:** 29.05.2010.

**Timing:** 60 min.

**Reference:** Chapter III Section 7 of Helgasson [Hel01].

In this talk we shall see that all real compacts form and even all Cartan decompositions are conjugated under the adjoint group. Prove also Exercice B.4.

## 1.6 Classical groups and their Cartan involutions

**Date:** 29.05.2010.

**Timing:** 60 min.

**Reference:** Chapter X Section 2 of Helgasson [Hel01].

In this talk we shall give example of Cartan decompositions for the classical groups and Lie algebras as well as examples of involutions of these groups and Lie algebras. Some statements at the beginning of subsection 2.3 in [Hel01] will be proved in the second session. The proof of Lemma 2.2 should be sketched.

## 2 Second session: June 12 2010

### 2.1 Orthogonal symmetric pairs

**Date:** 12.06.2010.

**Timing:** 60 min.

**Reference:** Chapter V Section 1 of Helgasson [Hel01].

In this talk we should define the orthogonal symmetric pairs and prove that any such pair can be decomposed in three different types, the compact, the non compact and the Euclidean ones.

### 2.2 Duality of orthogonal symmetric pairs

**Date:** 12.06.2010.

**Timing:** 60 min.

**Reference:** Chapter V Section 2 of Helgasson [Hel01].

In this talk we shall prove that compact and non compact orthogonal symmetric pairs are related by a duality preserving orthogonal symmetric pairs.

### 2.3 Further decompositions of orthogonal symmetric pairs

**Date:** 12.06.2010.

**Timing:** 60 min.

**Reference:** Chapter VIII Sections 5 in Helgasson [Hel01].

In this talk we shall prove that each orthogonal symmetric pair can be decomposed in one of four different types I, II, III and IV and describe how these types match with compact and non compact types. Do not deal with Proposition 5.5.

### 2.4 Cartan subalgebras and normal real forms

**Date:** 12.06.2010.

**Timing:** 60 min.

**Reference:** Chapter IX Sections 4 and 5 in Helgasson [Hel01].

In this talk we shall first define and study the analogue of Cartan subalgebras for real Lie algebras. We shall see that Cartan subalgebras are not always conjugated by that there are finitely many classes under conjugation. We shall also define another type of real form of a complex semisimple Lie algebras called normal real forms. These have special Cartan subalgebras (called split). We will prove that there is a unique real normal form.

Proofs of the affirmations in the beginning of Section 4 Chapter IX in [Hel01] should be given. Do not deal with Propositions 4.5 and 4.6. Do not deal also with Theorem 5.11.

## 2.5 Principle of the classification and a first alternative

**Date:** 12.06.2010.

**Timing:** 60 min.

**Reference:** Chapter X Section 1 in Helgasson [Hel01].

In this talk we shall recollect what we did in the previous talks and explain the principle of the classification of simple real Lie algebras. However, Proposition 1.4 should only be stated without proof. Prove the alternative given in Proposition 1.5.

## 2.6 Overview of the classification of finite automorphisms of complex semisimple Lie algebras

**Date:** 12.06.2010.

**Timing:** 60 min.

**Reference:** Chapter X Section 5 in Helgasson [Hel01].

In this talk we shall explain how to obtain the complete classification of real semisimple Lie algebras by describing all possible finite automorphisms of complex semisimple Lie algebras. We will also give a construction using the quaternion algebra  $\mathbb{H}$  of some classical simple real Lie algebras.

## References

- [Bou71] Bourbaki, N. *Éléments de mathématique. Fasc. XXVI. Groupes et algèbres de Lie. Chapitre I: Algèbres de Lie.* Seconde édition. Actualités Scientifiques et Industrielles, No. 1285 Hermann, Paris 1971.
- [Hel01] Helgason, Sigurdur *Differential geometry, Lie groups, and symmetric spaces.* Corrected reprint of the 1978 original. Graduate Studies in Mathematics, 34. American Mathematical Society, Providence, RI, 2001.
- [Ser66] Serre, Jean-Pierre *Algèbres de Lie semi-simples complexes.* W. A. Benjamin, inc., New York-Amsterdam 1966.

Lie Groups and Representation Theory Seminar. Organizers: Harry Tamvakis, Tom Haines. Abstract: We will talk about certain algebraic varieties, first studied by Kottwitz-Viehmann, that encodes orbital integrals of spherical Hecke functions on a reductive group over equal characteristic non-archimedean local field. We report on joint work with A. Bouthier that proves a dimension formula for these varieties and hence provides a description of the asymptotics of the corresponding orbital integrals.