Kirillov's Character Formula for Reductive Lie Groups

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Kirillov's famous formula says that the characters X of the irreducible unitary representations of a Lie group G should be given by an equation of the form

$$(\Phi) \qquad \chi(\exp x) = p(x)^{-1} \int_{\Omega} e^{i(\lambda,x)}\, d\mu_{\Omega}(\lambda)$$

where $\omega = \Omega(X)$ is a G-orbit in the dual $\mathfrak{g}^*$ f the Lie algebra $\mathfrak{g}$ of G, μ_Ω is Kirillov's canonical measure on Ω, and p is a certain function on $\mathfrak{g}$, namely $p(x) = \det^{1/2}\{\sinh(\mathrm{ad}(x/2))\,/\mathrm{ad}(x/2)\}$ at least for generic orbits Ω [10].
This formula cannot be taken too literally, of course (the integral in (Φ) is usually divergent), but has to be interpreted as an equation of distributions on a certain space of test functions on $\mathfrak{g}$. To make this precise, denote by $\mathfrak{g}^o$ an open neighborhoodod of zero in $\mathfrak{g}$ so that $\exp : \mathfrak{g} \to G$ restricts to an invertible analytic map of $\mathfrak{g}^o$ onto an open subset of G. For our purposes, the formula (Φ) should be interpreted as saying that

$$(\Phi') \qquad \mathrm{tr} \int_{\mathfrak{g}} \varphi(x)\pi(\exp(x))\, dx = \int_{\Omega} \int_{\mathfrak{g}} e^{i(\lambda,x)}\, \varphi(x)\, p(x)^{-1}\}\, d\mu_{\Omega}(\lambda)$$

for all C^∞ functions φ with compact support in $\mathfrak{g}^o$. (Here π is the representation of G with character χ.)
Of course, Kirillov's formula does not hold in this generality. It is in fact a major problem in representation theory to determine its exact domain of validity. In this paper we shall show that Kirillov's formula holds for the characters of a reductive real Lie group which occur in the Plancherel formula. Actually, we shall deal in detail only with the discrete series characters. The formula for the other characters can then be reduced to the formula for the discrete series characters by familiar methods. (Duflo [3]). Kirillov's formula for the discrete series is a consequence of a formula relating the Fourier transform on $\mathfrak{g}$ with the Fourier transform on Cartan subalgebras of compact type by means of the invariant integral. This is the form in which Kirillov's formula will be proved.

The proof depends of course heavily on the fundamental results of Harish- Chandra [5-7]. (These results are conveniently collected in Varadarajan's book [14], which will serve as standard reference for this paper.) In fact, it follows from Harish-Chandra's results that the characters in question are of the form

$$\chi(\exp x) = p(x)^{-1} \sum_\Omega c_\Omega \int_\Omega e^{i(\lambda,x)}\, d\mu_\Omega(\lambda)$$

where Ω runs over a finite set of orbits and the c_Ω's are complex constants. In this context the formula (Φ) simply says that c_Ω is in fact zero, except for a single orbit Ω for which it is one. The amount of effort and machinery involved in proving this simple assertion does seem somewhat surprising. An important ingredient in the proof is a Bochner type formula for the Fourier transform on a Euclidean space with indefinite metric, due to Strichartz [13].
Special cases of the result given here have been known for some time: the case when G is compact reduces essentially to Weyl's character formula together with results of Harish-Chandra and bas been worked out by Kirillov himself [10]; the case when G is complex semisimple by Gutkin [4]; and the case of the principal series of a real semisimple group by Duflo [3].
There is of course also the extensive literature on Kirillov's theory for nilpotent and solvable Lie groups, starting with Kirillov's original paper [9]. (Cf. [1,2], for example, for the solvable case.) Generalizations to other groups have been studied by Kirillov in [10], and by Lipsman in a recent paper [11], in which he also poses the problem of establishing Kirillov's formula for the characters of the discrete series of a semisimple Lie group.

We shall deal with the following kind of group, familiar from the work of Harish-Chandra (called "groups of class $\mathcal{H}$" in [14]):

(1) G is a real Lie group whose Lie algebra $\mathfrak{g}$ is reductive.
(2) G bas only finitely many connected components.
(3) $\mathrm{Ad}(G)$ is contained in $\mathrm{Int}(\mathfrak{g})$.
(4) The center of the connected subgroup with Lie algebra $[\mathfrak{g},\mathfrak{g}]$ is finite.

Recall that such a group bas a discrete series precisely when it bas a compact Cartan subgroup. This we assume to be so, and denote by T a fixed compact Cartan subgroup of G. The complexified Lie algebra $\mathfrak{t}_\mathbb{C}$ of T is then a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$ whose Weyl group will be denoted by $W_\mathbb{C}$. We write W for the subgroup $N_G(T)/T$ of $W_\mathbb{C}$. Fix once and for all a system of positive roots for $\mathfrak{t}_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}$ and denote by $\pi : \mathfrak{t}_\mathbb{C} \to \mathbb{C}$ the product of these positive roots. The set of regular elements in $\mathfrak{t}$ is then $\mathfrak{t}_r = \{t \in \mathfrak{t} \mid \pi(t) \neq 0\}$. If $\mathfrak{g}_e$ denotes the (open) set of regular, elliptic elements in $\mathfrak{g}$, then the map $G/T \times \mathfrak{t}_e \to \mathfrak{g}_e$, $(gT, t) \to g \cdot t$, onto, locally (analytically) invertible, and has fiber $\{(gwT, w^{-1}t)\}$ above $g \cdot t$.

If φ is a function on $\mathfrak{g}$, we denote by $M_\mathfrak{t}\varphi$ the (partially defined) function on $\mathfrak{t}$ given by

(1) $$\varphi(t) = \pi(t) \int_G \varphi(g \cdot t)\, dg$$

whenever this integral converges. The integral here is taken with respect to Haar measure on G normalized so that ([14], Lemma 2, p. 37):

(2) $\int_{\mathfrak{g}_o} \varphi(x)\, dx = \int_{\mathfrak{t}} |\pi(t)|^2 \{\int_G \varphi(g\cdot t)\, dg\}\, dt.$

when the Lebesgue measures on $\mathfrak{g}$ and on $\mathfrak{t}$ are normalized as indicated below. From the regularity properties of the map $G/T \times \mathfrak{t}_e \to \mathfrak{g}_e$ it is clear that $M_{\mathfrak{t}}$ maps the space $D(\mathfrak{g}_e)$
compactly supported C^∞ functions on $\mathfrak{g}_e$ (with its usual topology) continuously onto the space $D(\mathfrak{t}_r)$of W-anti-invariant compactly supported C^∞ functions on $\mathfrak{t}_r$. (A function φ on $\mathfrak{t}$ is W-anti-invariant if $\tau(w)\varphi = \epsilon(w)\varphi$ for all $w \in W$; here $\epsilon(w) = \det(w : \mathfrak{t} \to \mathfrak{t})$.)

We fix a G-invariant inner product (,) on $\mathfrak{g}$ which is negative definite on $\mathfrak{t}$ and use it to define the Fourier transforms on $\mathfrak{g}$ and on $\mathfrak{t}$:

(3) $F_{\mathfrak{g}} = \int_{\mathfrak{g}} e^{i(x,y)}\, \varphi(y)\, dy,$

(4) $F_{\mathfrak{t}} = \int_{\mathfrak{t}} e^{i(t,s)}\, \varphi(s)\, ds.$

The Lebesgue measures on $\mathfrak{g}$ and on $\mathfrak{t}$ we assume normalized so that $F_{\mathfrak{g}}^2\varphi(x) = \varphi(-x)$ and $F_{\mathfrak{g}}^2\varphi(t) = \varphi(-t)$ for rapidly decreasing functions φ. We also extend $F_{\mathfrak{g}}$ and $F_{\mathfrak{t}}$ transforms of tempered distributions in the usual way.
Set $\widehat{D}(\mathfrak{g}_e) = F_{\mathfrak{g}} D(\mathfrak{g}_e), \widehat{D}(\mathfrak{t}_r) = F_{\mathfrak{t}} D(\mathfrak{t}_r)$, and write $\widehat{D}(\mathfrak{t}_r)_W = F_{\mathfrak{t}} D(\mathfrak{t}_r)_W$ for the W-anti- invariants in $\widehat{D}(\mathfrak{t}_r)$. With this notation the main result of this paper can be stated
as follows.

Theorem. *$M_{\mathfrak{t}}$ maps $D(\mathfrak{g}_e)$ onto $D(\mathfrak{t}_r)$and $\widehat{D}(\mathfrak{g}_e)$ onto $\widehat{D}(\mathfrak{t}_r)$. This map $M_{\mathfrak{t}}$ satisfies*

$$M_{\mathfrak{t}} F_{\mathfrak{g}}\varphi = \gamma F_{\mathfrak{t}} M_{\mathfrak{t}}\varphi$$

for $\varphi \in D(\mathfrak{g}_r)$ and all $\varphi \in \widehat{D}(\mathfrak{g}_r)$. Here

$$\gamma = (i)^{\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{t})}(-1)^{\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{k})}$$

where $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$ for which $\mathfrak{k}\cap[\mathfrak{g},\mathfrak{g}]$ is maximal compactly embedded in $[\mathfrak{g},\mathfrak{g}]$. (So $\dim(\mathfrak{g}/\mathfrak{k})$ is the maximal dimension of a subspace of $\mathfrak{g}$ on which (,) is positive definite.)

The first step in the proof of this theorem is the following lemma, which is an elaboration on a result of Harish-Chandra ([14J Thm. 7, p. 111).

Lemma A. *There is a function $c : W \to \mathbb{C}$ so that*

$$M_{\mathfrak{t}} F_{\mathfrak{g}}\varphi = (-1)^{\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{k})} \sum_{w\in W_{\mathbb{C}}} c(w)\, \tau(w)\, F_{\mathfrak{t}} M_{\mathfrak{t}}\varphi$$

for all $\varphi \in D(\mathfrak{g}_e)$ and so that

(1) $c(w) = c(w^{-1})$ *for all* $w \in W_{\mathbb{C}}$,
(2) $c(vw) = \epsilon(v)c(w)$ *for all* $w \in W_{\mathbb{C}}$,
(3) $c * \widehat{c} = \epsilon_W = \widehat{c} * c$ *for some function* $\widehat{c} : W_{\mathbb{C}} \to \mathbb{C}$ *which also satisfies* (1) *and* (2). *Here*

$$c * c \wedge (w) = \sum_{v\in W_{\mathbb{C}}} c(wv^{-1})c(v), \qquad \epsilon_W(w) = \begin{cases} \epsilon(w) & \text{if } w \in W \\ 0 & \text{if } w \notin W \end{cases}.$$

To prove this lemma we introduce for $t \in \mathfrak{t}_r$ the (tempered) distribution μ_t on $\mathfrak{g}$ defined by

(5) $\quad (\mu_r, \varphi) = \pi(t) \int_G \varphi(g \cdot t)\, dg$

and denote by $\theta_t = F_{\mathfrak{g}} \mu_t$ its Fourier transform. According to a theorem of Harish- Chandra θ_t is a locally integrable function whose restriction to tr is given by the formula

(6) $\quad \theta_t(s) = \pi^{-1}(s) \sum_{w \in W_{\mathbb{C}}} c(w,t)\, e^{i(w \cdot t, s)}$

where for fixed $w \in W$, $t \mapsto c(w,t)$ is a locally constant function on $\mathfrak{t}_r$ ([14 Thm.7], p.111). Now these functions are actually constant on all of $\mathfrak{t}$. To see this we show that

(7) $\quad \pi(s)\theta_t(s) = \pi(t)\theta_s(t)$

for all s, t in $\mathfrak{t}_r$. This will be sufficient because in view of (6) the equation(7) says that

$$\sum_{w \in W_{\mathbb{C}}} c(w,t)\, e^{i(w \cdot t, s)} = \sum_{w \in W_{\mathbb{C}}} c(w,s)\, e^{i(w \cdot s, t)}$$
$$= \sum_{w \in W_{\mathbb{C}}} c(w^{-1}, s)\, e^{i(w \cdot t, s)}$$

for all s, t in $\mathfrak{t}_r$. But for fixed $t \in \mathfrak{t}_r$ the functions $s \to e^{i(wt,s)}, w \in W_{\mathbb{C}}$, are linearly independent (even after restriction to a connected component of $\mathfrak{t}_r$, where $s \to c(w,s)$ is constant). So $c(w,t) = c(w^{-1}, s)$ for all s, t in $\mathfrak{t}_r$, $c(w,t) = c(w)$ is constant and $c(w) = c(w^{-1})$. To prove (7) it suffices to show that

(8) $\quad \int_{\mathfrak{t} \times \mathfrak{t}} \pi(s)\theta_t(s) M_{\mathfrak{t}}\varphi(s)\, M_{\mathfrak{t}}\psi(t)\, ds\, dt = \int_{\mathfrak{t} \times \mathfrak{t}} \pi(t)\theta_s(t) M_{\mathfrak{t}}\varphi(s)\, M_{\mathfrak{t}}\psi(t)\, ds\, dt$

for all $\varphi, \psi \in D(\mathfrak{g}_e)$ (because $\pi(s)\theta_t(s)$ and $\pi(t)\theta_t(t)$ are W-anti-invariant in s and t, and $\mathrm{M}_{\mathfrak{t}}$ maps $D(\mathfrak{g}_e)$ onto $D(\mathfrak{t}_r)_W$). For this we use formula (2) and the fact that $|\pi(t)|^2 = (-1)^{\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{t})} \pi(t)^2$ to compute

$$\begin{aligned}
&\int_{\mathfrak{t} \times \mathfrak{t}} \pi(s)\theta_t(s) M_{\mathfrak{t}}\varphi(s)\, M_{\mathfrak{t}}\psi(t)\, ds\, dt = \\
&\quad = \int_{\mathfrak{t}} \{ \int_{\mathfrak{t}} \int_G \pi(s)^2 \theta_t(s) \varphi(g \cdot s)\, dg\, ds \}\, M_{\mathfrak{t}}\psi(t)\, dt \\
&\quad = (-1)^{\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{t})} \int_{\mathfrak{t}} \{ \int_{\mathfrak{g}} \theta_t(x) \varphi(x)\, dx \}\, M_{\mathfrak{t}}\psi(t)\, dt \\
&\quad = (-1)^{\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{t})} \int_{\mathfrak{t}} \{ \pi(t) \int_G \varphi(g \cdot t)\, dg \}\, M_{\mathfrak{t}}\psi(t)\, dt \\
&\quad = \int_{\mathfrak{t}} \int_G \int_G F_{\mathfrak{g}}\varphi(g \cdot t)\, dg\, \psi(h \cdot t)\, |\pi(t)|^2\, dg\, dh\, dt \\
&\quad = \int_G \int_{\mathfrak{t}} \int_G F_{\mathfrak{g}}\varphi(g \cdot t)\, \psi(h \cdot t)\, |\pi(t)|^2\, dh\, dt\, dg \\
&\quad = \int_G \int_{\mathfrak{g}} F_{\mathfrak{g}}\varphi(g \cdot x)\, \psi(x)\, dx\, dg \\
&\quad = \int_G \int_{\mathfrak{g} \times \mathfrak{g}} e^{i(g \cdot x, y)} \varphi(y)\, \psi(x)\, dx\, dy\, dg\,.
\end{aligned}$$

Since this expression is symmetric in φ and ψ we get (8).
Next we show that the function $c : W_{\mathbb{C}} \to \mathbb{C}$ defined in this way satisfies the conditions (1) – (3) of the lemma. (1) we already know. For (2) we note that $\mu_{v \cdot t} = \epsilon(v)\mu_t$ for all $v \in W$, which is immediate from (5). Therefore

$$\theta_{v \cdot t}(s) = \epsilon(v)\pi(s)^{-1} \sum_{w \in W_{\mathbb{C}}} c(w)\, e^{i(w \cdot t, s)}.$$

But also

$$\theta_{v\cdot t}(s) = \pi(s)^{-1}\sum_{w\in W_{\mathbb{C}}} c(w)\, e^{i(wv\cdot t,s)}$$
$$= \pi(s)^{-1}\sum_{w\in W_{\mathbb{C}}} c(wv^{-1})\, e^{i(w\cdot t,s)}.$$

Comparison of the formulas gives that that $c(wv^{-1}) = \epsilon(v)c(w)$ for all $v \in W$ and $w \in W_{\mathbb{C}}$, hence also $c(vw) = \epsilon(v)c(w)$ in view of (1). To prove (3) we use the fact that the fact that the matrix $[c(uv^{-1})\}$, $u, v \in W_{\mathbb{C}}$, has rank $|W_{\mathbb{C}}/W|$ ([14], Thm.20, p.121), which is precisely the dimension of the space of functions $a : W_{\mathbb{C}} \to \mathbb{C}$ satisfying $a(vw) = \epsilon(v)a(w)$ for all $v \in W$, $w \in W_{\mathbb{C}}$. So the map $a \to c * a$ defines a linear automorphism of this space of functions. In particular there is a unique such function $\widehat{c}$ so that $c * \widehat{c} = \epsilon_W$. This $\widehat{c}$ clearly also satisfies (1).

Finally it remains to be shown that this function c satisfies the first assertion of the lemma. For this we use formula (2) to compute:

$$M_{\mathfrak{t}}F_{\mathfrak{g}}\varphi(t) = (\mu_t, F_{\mathfrak{g}}\varphi)$$
$$= \int_{\mathfrak{t}}\int_G \theta_t(g\cdot s)\,\varphi(g\cdot s)\,dg\,|\pi(s)|^2\,dg\,ds$$
$$= (-1)^{\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{t})}\int_{\mathfrak{t}} \pi(s)\,\theta(s)\,\{\pi(s)\int_G \varphi(g\cdot s)\,gd\}\,ds$$
$$= (-1)^{\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{t})}\sum_{w\in W_{\mathbb{C}}} c(w)\int_{\mathfrak{t}} e^{i(w\cdot t,s)}\,M_{\mathfrak{t}}\varphi(s)\,ds$$
$$= (-1)^{\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{t})}\sum_{w\in W_{\mathbb{C}}} c(w)F_{\mathfrak{t}}\,M_{\mathfrak{t}}\varphi(w\cdot t).$$

Write $D'(\mathfrak{g}_e)$ for the topological dual of $D(\mathfrak{g}_e)$ (i.e. for the space of distributions on $\mathfrak{g}_e$) and $\widehat{D}'(\mathfrak{g}_e)$ for the dual of $D'(\mathfrak{g}_e)$. The map $F_{\mathfrak{g}} : D'(\mathfrak{g}_e) \to \widehat{D}\,'(\mathfrak{g}_e)$ gives $F'_{\mathfrak{g}} : \widehat{D}\,'(\mathfrak{g}_e) \to D'(\mathfrak{g}_e)$. (I write A' for the transpose of a continuous map A between topological vector spaces.) $F'_{\mathfrak{g}}$ clearly clearly maps $F_{\mathfrak{g}} : \widehat{D}'(\mathfrak{g}_e)^G$ onto $\widehat{D}\,'(\mathfrak{g}_e)^G$) (the superscript G denoting the G-invariants.)

Similarly we have a map $F'_{\mathfrak{t}} : \widehat{D}\,'(\mathfrak{t}_r) \to D'(\mathfrak{t}_r)$ mapping $\widehat{D}\,'(\mathfrak{t}_r)_W$ onto $D'(\mathfrak{t}_r)_W$ (the subscript W denoting W-anti-invariants). Finally, the map $M_{\mathfrak{t}} : D(\mathfrak{g}_e) \to D(\mathfrak{t}_r)_W$ gives rise to a map $M'_{\mathfrak{t}} : D'(\mathfrak{t}_r)_W \to D'(\mathfrak{g}_e)^G, \widehat{D}\,'(\mathfrak{t}_r)_W \to \widehat{D}\,'(\mathfrak{g}_e)^G$. Note that $\varphi \in D'(\mathfrak{t}_r)_W$ is a function, so is $M_{\mathfrak{t}}\varphi \in \widehat{D}\,'(\mathfrak{g}_e)^G : M_{\mathfrak{t}}\varphi(x) = \pi(t)^{-1}\varphi(t)$ if $x = g\cdot t$ with $g \in G$ and $t \in \mathfrak{t}_r$. The next lemma shows how certain distributions behave under these maps:

Lemma B. (1) *If h is a W-anti-invariant, harmonic polynomial on $\mathfrak{t}$, then $M'_{\mathfrak{t}}h$ extends to a G-invariant, harmonic, tempered distribution H on $\mathfrak{g}$. If h is homogeneous of degree* $\deg(h)$*, then H is homogeneous of degree* $\deg(h) - \frac{1}{2}\dim(\mathfrak{g}/\mathfrak{t})$.
(2) *For h as above*

$$F'_{\mathfrak{g}}M'_{\mathfrak{t}}(rh) = (i)^{\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{t})}(-1)^{\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{k})}M'_{\mathfrak{t}}F'_{\mathfrak{t}}(rh)$$

on $\mathfrak{g}_e$,for every function r of the form $r(t) = f(|t|)$, $f \in C_c^\infty((0,\infty))$.

By "harmonic" is meant "annihilated by the Laplacian $\Delta_{\mathfrak{g}}$ or $\Delta_{\mathfrak{t}}$ of the inner product (,) on $\mathfrak{g}$ or on $\mathfrak{t}$"; and $|t| = |(t,t)|^{1/2}$.

The assertions of the lemma should be understood as follows. In (1) we think of h as an element of $D'(\mathfrak{t}_r)_W$ so that $M'_{\mathfrak{t}}h$ in $D'(\mathfrak{g}_e)^G$. In (2) we think of rh as an element of $\widehat{D}'(\mathfrak{t}_r)_W$ so that $F'_{\mathfrak{g}}M'_{\mathfrak{t}}(rh)$ and $M'_{\mathfrak{t}}F'_{\mathfrak{t}}(rh)$ are both in $D'(\mathfrak{g}_e)^G$.

To prove this lemma we denote by h for the moment any W-anti-invariant polynomial on $\mathfrak{t}$. Write D_h the constant coefficient operator corresponding to multiplication by h under $F_\mathfrak{t}$. Now, according to a result of Harish-Chandra ([14] Cor. 24, p. 50), the formula

$$\varphi \to \lim_{t\to 0^+} D_h M_\mathfrak{t}\varphi(t)$$

defines a tempered distribution on $\mathfrak{g}$. (Here $t \to 0^+$ means that t goes to zero in the positive Weyl chamber. Any other Weyl chamber would also do.) We can therefore define a tempered distribution ν^h on $\mathfrak{g}$ by setting

$$(\nu^h, \varphi) = \frac{(-1)^{\dim(\mathfrak{g}/\mathfrak{t})}}{|W|} \sum_{w\in W_\mathbb{C}} \widehat{c}(w) \lim_{t\to 0^+} D_h \tau(w) M_\mathfrak{t}\varphi(t).$$

Let $H = F_\mathfrak{g}\nu^h$ be its Fourier transform. Then H is a tempered, G-invariant distribution on $\mathfrak{g}$, and I claim that H coincides with $M'_\mathfrak{t}h$ on $\mathfrak{g}_e$. In fact, if $\varphi \in D(\mathfrak{g}_e)$, then

$$\begin{aligned}(H,\varphi) &= (\nu^h, F_\mathfrak{g}\varphi)\\ &= \frac{(-1)^{\dim(\mathfrak{g}/\mathfrak{t})}}{|W|} \sum_{w\in W_\mathbb{C}} \widehat{c}(w) \lim_{t\to 0^+} D_h \tau(w) M_\mathfrak{t} F_\mathfrak{g}\varphi(t).\\ &= \frac{1}{|W|} \sum_{u,v\in W_\mathbb{C}} \widehat{c}(u)c(v)\, D_h \tau(u)\tau(v) F_\mathfrak{t} M_\mathfrak{t}\varphi(0).\end{aligned}$$

(by Lemma A. – The limit becomes evaluation at $t = 0$ because $F_\mathfrak{t}M_\mathfrak{t}\varphi$, $\varphi \in \widehat{D}(\mathfrak{t}_r)_W$ is C^∞ on al of $\mathfrak{t}$). Thus

$$\begin{aligned}(H,\varphi) &= (\nu^h, F_\mathfrak{g}\varphi)\\ &= \frac{1}{|W|} \sum_{u,v\in W_\mathbb{C}} \widehat{c}(u)c(v)\, D_h \tau(u)\tau(v) F_\mathfrak{t} M_\mathfrak{t}\varphi(0)\\ &= \frac{1}{|W|} \sum_{w\in W_\mathbb{C}} \widehat{c} * c(w)\, D_h \tau(w) F_\mathfrak{t} M_\mathfrak{t}\varphi(0)\\ &= \frac{1}{|W|} \sum_{w\in W_\mathbb{C}} \epsilon(w)\, D_h \tau(w) F_\mathfrak{t} M_\mathfrak{t}\varphi(0)\\ &= D_h F_\mathfrak{t} M_\mathfrak{t}\varphi(0)\\ &= F_\mathfrak{t} h M_\mathfrak{t}\varphi(0)\\ &= (h, M_\mathfrak{t}\varphi)\\ &= (M'_\mathfrak{t}h, \varphi).\end{aligned}$$

So H does indeed coincide with $M'_\mathfrak{t}h$ on $\mathfrak{g}_e$. Moreover, if h is homogeneous of degree $\deg(h)$, then H is clearly homogeneous degree $\deg(h) - \deg(\pi) = \deg(h) - \frac{1}{2}\dim(\mathfrak{g}/\mathfrak{t})$.

To prove that H is harmonic when h is we argue as follows. Let P be any G-invariant polynomial on $\mathfrak{g}$, p its restriction to $\mathfrak{t}$, and D_P (resp. D_p) the corresponding constant coefficient operators on $\mathfrak{g}$ {resp. on $\mathfrak{t}$). For any rapidly decreasing C^∞ function φ on $\mathfrak{g}$

$$\begin{aligned}(D_P H,\varphi) &= (D_P F_\mathfrak{g}\nu^h, \varphi)\\ &= (F_\mathfrak{g}\nu^h, P\varphi)\\ &= \frac{(-1)^{\dim(\mathfrak{g}/\mathfrak{t})}}{|W|} \sum_{w\in W_\mathbb{C}} \widehat{c}(w) \lim_{t\to 0^+} D_h \tau(w) M_\mathfrak{t} P F_\mathfrak{g}\varphi(t)\\ &= \frac{(-1)^{\dim(\mathfrak{g}/\mathfrak{t})}}{|W|} \sum_{w\in W_\mathbb{C}} \widehat{c}(w) \lim_{t\to 0^+} D_h \tau(w) p M_\mathfrak{t} F_\mathfrak{g}\varphi(t).\end{aligned}$$

Now one easily verifies that for any two polynomials p, q on a Euclidean space and any C^∞ function φ one has

$$D_q(p\varphi) = D_{q'}\varphi + \psi$$

where $q' = D_p q$ and ψ is a function (defined by this equation) which satisfies $\psi(0) = 0$. We would like to apply this observation to the terms $D_h\tau(w)pM_{\mathfrak{t}}F_{\mathfrak{g}}\varphi(t)$ in the above sum. Now $M_{\mathfrak{t}}F_{\mathfrak{g}}\varphi$ need not be C^∞ on all of $\mathfrak{t}$, but its partials do extend to continuous functions on the closure of any fixed Weyl chamber ([14J Thm.23, p. 50). From this one can conclude that the function ψ on $\mathfrak{t}_r$ defined by

$$D_h pM_{\mathfrak{t}}F_{\mathfrak{g}}\varphi = D_{h'}M_{\mathfrak{t}}F_{\mathfrak{g}}\varphi + \psi \quad (\text{on } \mathfrak{t}_r),\ h' = D_p h$$

still bas the property that $\psi(t) \to 0$ as $t \to 0^+$. (By subtracting a suitable polynomial from $M_{\mathfrak{t}}F_{\mathfrak{g}}\varphi$ it evidently suffices to see that a function f on $\mathfrak{t}_r$ whose partials up to sufficiently high order all vanishes as $t \to 0^+$ has the property that $D_h(pf) = D_{h'}f + g$ where $g(t) \to 0$ as $t \to 0^+$. But this is clear, since – for fixed $h, p-$ g is a linear expression in the partials of f with polynomial functions for coefficients.) So the formula for D_pH becomes

$$(D_PH, \varphi) = \text{const.} \sum_{w\in W_{\mathbb{C}}} \widehat{c}(w)\ \lim_{t\to 0^+} D_{h_w}\tau(w)M_{\mathfrak{t}}F_{\mathfrak{g}}\varphi(t).$$

where $h_w = D_{\tau(w)p}h$. In particular, for $P(x) = (x, x)$ we have $p(t) = (t, t)$, $D_P = \Delta_{\mathfrak{g}}$, $D_p = \Delta_{\mathfrak{t}}$. So if h is harmonic, then $h_w = 0$ for all $w \in W_{\mathbb{C}}$ and consequently $(\Delta_{\mathfrak{g}}H, \varphi) = 0$ also.

This proves the first part of the lemma. For the second part we need Strichartz's extension of Bochner's formula for the Fourier transform of a distribution of the type "radial× homogeneous harmonic" for Euclidean spaces with indefinite metric [13]. We therefore introduce the following notation

$E = \mathbb{R}^n$ with inner product
$(x, x) = x_1^2 + \cdots + x_{n^+}^2 - x_{n^++1}^2 - \cdots - x_{n^++n^-}^2$, $(n^+ + n^- = n)$,
$|x| = |(x, x)|^{1/2}$
$E_\pm = \{x \in E \mid \pm(x, x) > 0\}$
$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{n_+}^2} - \cdots - \frac{\partial^2}{\partial x_n^2}$.

We define the Fourier transform F on E by $F\varphi(x) = \int_E e^{i(x,x)}\,\varphi(y)\,dy$ and assume the Lebesgue measure on E normalized so that $F^2\varphi(x) = \varphi(-x)$. We let $\mathbb{R}^\times$ operate on functions on E by setting $\tau(a)\varphi(x) = \varphi(a^{-1}x)$ for $a \in \mathbb{R}^\times$ and $\varphi : E \to \mathbb{C}$. We extend this action to distributions on E by requiring that $(\tau(a)\varphi, \tau(a)\psi) = |a|^n(\varphi, \psi)$ for distributions φ and test functions ψ. With this notation we can state:

Strichartz's Formula. Let φ be a tempered distribution on E which satisfies $\Delta\varphi$=0, $\tau(a)\varphi = |a|^{-\sigma}\text{sgn}^\epsilon(a)\varphi$ for some $\sigma \in \mathbb{C}$ and $\epsilon = 0, 1$ (identically in $a \in \mathbb{R}^\times$). Let ψ be a functions supported on $E_\pm$ of the form $\psi(x) = f(|x|)$ for $x \in E_\pm$ with $f \in C_c^\infty((0, \infty))$. Then the restriction to $E_\pm$ of the Fourier transform of the distribution $\psi\varphi$ is given by the formula $F(\psi\varphi) = \tilde{\psi}\varphi$ (on E) where $\tilde{\psi}$ is the function on $E_\pm$ defined by

$$\tilde{\psi}(x) = i^\epsilon \int_0^\infty \{\cos\tfrac{\pi}{2}(n^\pm + \sigma + \epsilon)J_\alpha(|x|t) \\ - \sin\tfrac{\pi}{2}(n^\pm + \sigma + \epsilon)Y_\alpha(|x|t)\}f(t)|x|^{-\alpha}t^{\alpha+1}\,dt$$

Here $\alpha = n/2 + \sigma - 1$ and J_α, Y_α are the usual special functions denoted by these symbols. The assertions are to be taken in the sense that either the upper sign or the lower sign in the symbols $\pm$ is consistently chosen throughout. The formula differs from the one in [13] by a factor of $(2\pi)^{n/2}$ because of different normalizations of the Lebesgue measure on E. Strichartz also requires that φ be a C^∞ function on $E_\pm$, but this superfluous since on $E_\pm$ a homogeneous harmonic distribution is a limit (in the distribution sense) of homogeneous harmonic C^∞ functions (in fact of harmonic analytic functions, as one sees from Lemma 3(a) of [12]).

We apply this formula as follows. First we take $E = \mathfrak{t}$ with its inner product (,). For φ we take a homogeneous harmonic polynomial h on $\mathfrak{t}$ and for ψ a function $r(t) = f(|t|)$ as in the lemma. Then on $\mathfrak{t}\backslash\{0\}$, $F_\mathfrak{t}(rh) = \tilde{r}h$ where $\tilde{r}$ is the function defined at $x \in \mathfrak{t}\backslash\{0\}$ by taking the rower signs on the rhs of (10) and substituting

$$n^+ = 0;\, \sigma = \deg(h);\, \epsilon \equiv \deg(h) \bmod 2;\, \alpha = \tfrac{1}{2}\dim(\mathfrak{t}) + \deg)h) - 1.$$

This gives (for $x \in \mathfrak{t}\backslash\{0\}$):

$$\tilde{r}(x) = (-i)^{\deg(h)} \int_0^\infty J_\alpha(|x|)\, f(t)\, |x|^{-\alpha} t^{\alpha+1}\, dt$$

Next we take $E = \mathfrak{g}$ with its inner product (,). For φ we take a harmonic distribution H (homogeneous of degree $\deg(h) - \frac{1}{2}\dim(\mathfrak{g}/\mathfrak{t})$) as in part (1) of the lemma, and for ψ the function R on $E_- = \mathfrak{g}_-$ defined by $R(x) = f(|x|)$ (f as above). Then on $\mathfrak{g}_-$, $F_\mathfrak{g}(RH) = \tilde{R}H$, where $\tilde{R}$ is the function defined at $x \in \mathfrak{g}_-$ by taking the lower signs on the rhs of (10) and substituting

$$n^+ = \dim(\mathfrak{g}/\mathfrak{t});\, \sigma = \deg(h) - \tfrac{1}{2}\dim(\mathfrak{g}/\mathfrak{t}),$$
$$\epsilon = \deg(h) - \tfrac{1}{2}\dim(\mathfrak{g}/\mathfrak{t}) \bmod 2, \quad \alpha = \tfrac{1}{2}\dim(\mathfrak{t}) + \deg(h) - 1.$$

This gives (for $x \in \mathfrak{g}_-$):

$$\tilde{R}(x) = \gamma(-i)^{\deg(h)} \int_0^\infty J_\alpha(|x|)\, f(t)\, |x|^{-\alpha} t^{\alpha+1}\, dt$$

where $\gamma = (i)^{\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{t})}(-1)^{\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{k})}$.

Since α has the same value for $\mathfrak{g}$ and for $\mathfrak{t}$ we find that $\tilde{R} = \gamma\tilde{r}$ on $\mathfrak{t}\backslash\{0\}$. We now use the fact that $\mathfrak{g}$ is a subset of $\mathfrak{g}_-$ and the relations

$$M'_\mathfrak{t}(rh) = RM_{\mathfrak{t}'}(h) = RH \quad (\text{on } \mathfrak{g}_e)$$
$$M'_\mathfrak{t}(\tilde{r}h) = \gamma^{-1}\tilde{R}M_{\mathfrak{t}'}(h) = \gamma^{-1}\tilde{R}H \quad (\text{on } \mathfrak{g}_e)$$

to compute

$$\text{(11)} \qquad \begin{aligned} M'_\mathfrak{t}(rh) &= M'_\mathfrak{t}(\tilde{r}h) \quad (\text{on } \mathfrak{g}_e) \\ &= \gamma^{-1}\tilde{R}H \quad (\text{on } \mathfrak{g}_e) \\ &= \gamma^{-1}F'_\mathfrak{g}RH \quad (\text{on } \mathfrak{g}_e) \\ &= \gamma^{-1}F'_\mathfrak{g}M'_\mathfrak{t}(rh) \quad (\text{on } \mathfrak{g}_e) \end{aligned}$$

This finishes the proof of Lemma B.

To complete the proof of the Theorem we compare Lemma A with Lemma B to find that (notation as above, $\delta = (-1)^{\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{t})}$):

(12) $\quad M_{\mathfrak{t}}' F_{\mathfrak{t}}'(rh) = \frac{\delta}{|W|} \sum_{w \in W_{\mathbb{C}}} \widehat{c}(w)\, F_{\mathfrak{g}}' M_{\mathfrak{t}}' \tau(w)'(rh)$

(by taking transposes in Lemma A); but also

(13) $\quad M_{\mathfrak{t}}' F_{\mathfrak{t}}'(rh) = \gamma^{-1} F_{\mathfrak{g}}' M_{\mathfrak{t}}'(rh)$

where (12) and (13) are understood as equations between distributions on $\mathfrak{g}_e$. Now by (11) these distributions area actually functions on $\mathfrak{g}_e$, namely

$$M_{\mathfrak{t}}' F_{\mathfrak{t}}'(rh) = \gamma^{-1} \tilde{R} H, \qquad F_{\mathfrak{g}}' M_{\mathfrak{t}}'(rh) = \tilde{R} H.$$

So if we equate the rhs of (12) and (13) and restrict to $\mathfrak{t}_r$ we get

$$\frac{\gamma\delta}{|W|} \sum_{w \in W_{\mathbb{C}}} \widehat{c}(w) \tau(w) \tilde{r} h = \tilde{r} h$$

Since r radial (hence $W_{\mathbb{C}}$-invariant) this shows that

$$\frac{\gamma\delta}{|W|}\ \tilde{r}\ \sum_{w \in W_{\mathbb{C}}} \widehat{c}(w) \tau(w) h = \tilde{r} h$$

as function on $\mathfrak{t}_r$. Thus

$$\frac{\gamma\delta}{|W|} \sum_{w \in W_{\mathbb{C}}} \widehat{c}(w) \tau(w) h = h$$

for all W-anti-invariant harmonic polynomials h on $\mathfrak{t}$. Since every polynomial on $\mathfrak{t}$ is a sum of polynomials of the form $p(x) = g((x,x))h)x)$ with $h(x)$ harmonic we see that (14) holds for all W-anti-invariant polynomials h on $\mathfrak{t}$. So the operator $\frac{\gamma\delta}{|W|} \sum_{w \in W_{\mathbb{C}}} \widehat{c}(w) \tau(w)$ maps the polynomials on $\mathfrak{t}$ onto the W-anti-invariant polynomials and leaves the W-anti-invariants fixed. This means that this operator is the W-projection onto the space of W-anti-invariants i.e.

$$\frac{\gamma\delta}{|W|} \sum_{w \in W_{\mathbb{C}}} \widehat{c}(w) \tau(w) = \qquad \frac{1}{|W|} \sum_{w \in W} \epsilon(w) \tau(w).$$

Comparing coefficients shows that

$$\widehat{c}(w) = \begin{cases} \frac{1}{\gamma\delta} \epsilon(w) & \text{if } \ w \in W \\ 0 & \text{if } \ w \notin W \end{cases}$$

Equation (3) of Lemma A now gives

$$c(w) = \begin{cases} \frac{\gamma\delta}{|W|} \epsilon(w) & \text{if } \ w \in W \\ 0 & \text{if } \ w \notin W \end{cases}.$$

Substituting this formula for $c(w)$ into the equation in Lemma A completes the proof of the Theorem. As a consequence of the theorem we get:

Corollary. *The formula $(R_{\mathfrak{t}}\varphi, M_{\mathfrak{t}}\varphi) = (\varphi, \psi)$ defines a linear isomorphism $R_{\mathfrak{t}}$ of $D'(\mathfrak{g}_e)^G$ onto $D'(\mathfrak{t}_t)_W$ and of $\widehat{D}'(\mathfrak{g}_e)^G$ onto $\widehat{D}'(\mathfrak{t}_t)_W$. This map satisfies*
$R_{\mathfrak{t}} F_{\mathfrak{g}}' = \gamma F_{\mathfrak{t}} R_{\mathfrak{t}}$.

The fact that $R_{\mathfrak{t}}$ is an isomorphism of $D'(\mathfrak{g}_e)^G$ onto $D'(\mathfrak{t}_t)_W$ is a well-kown consequence of the regularity properties of the map $G \times \mathfrak{T}_r \to \mathfrak{g}_e$ ([14J Thm.3, p. 25). The other assertions then follow from the theorem. Another Consequence of the theorem is the following:

Corollary. *The Fourier transform of the distribution μ_t ($t \in \mathfrak{t}_r$) on $\mathfrak{g}$ defined by*

(15) $\quad (\mu_t, \varphi) = \int_G \varphi(g \cdot t)\, dg$

is the unique tempered, G-invariant eigendistribution (necessarily a function) whose restriction to $\mathfrak{t}_r$ is given by

$$\theta_t(s) = \kappa\pi(s)^{-1} \sum_{w\in W} \epsilon(w)\, e^{i(w\cdot s,t)}, \quad \kappa = (-i)^{\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{t})}(-1)^{\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{k})}\, |W|^{-1}$$

This is immediate from equations (6)and (15) together with a we -known result of Harish-Chandra ([14] Thm.,16, p. 119).

The significance of the distributions θ_t lies in the fact that they are essentially the discrete series characters in exponential coordinates. To make this precise, let $\mathfrak{c}^0$ be an open neighborhood of zero in the center of $\mathfrak{g}$ on which $\exp : \mathfrak{c}^0 \to G$ is an invertible an invertible analytic map auto its image. Write $[\mathfrak{g}, \mathfrak{g}]^0$ for the set of elements x of $[\mathfrak{g}, \mathfrak{g}]$ for which the eigenvalues λ of $\mathrm{ad}(x)$ satisfy $|\mathrm{Im}(\lambda)| < \pi$ and set $\mathfrak{g}^0 = \mathfrak{c}^0 + [\mathfrak{g}, \mathfrak{g}]^0$. $\mathfrak{g}^0$ is an open neighborhood of zero in $\mathfrak{g}$ and $\exp : \mathfrak{g}^0 \to G$ is an invertible analytic map onto its image ([14] Cor. 6, p. 194). This map allows us to identify distributions on $\exp(\mathfrak{g}^0)$ with distributions on $\mathfrak{g}^0$.
Next, let $\tau : T \to \mathbb{C}$ an irreducible character of $T$and write the differential of τ in the form $d\tau : \mathfrak{t} \to \mathbb{R}$, $s \mapsto \deg(\tau)(it - r, s)$, where r is the half-sum of the positive roots. Assume that τ; is regular, i.e. that t is in $\mathfrak{t}_r$. $\deg(\tau)$ is the degree of τ (which may be > 1, since T need not be abelian). According to Harish- Chandra's construction there is a discrete series character Θ_τ associated to τ which satisfies

(16) $\quad \Theta_\tau(\exp(x)) = \deg(\tau)\, (i)^{\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{t})}|W|\epsilon(t)p(x)^{-1}\theta_t(x)$

as an identity of distributions on $\mathfrak{g}^0$. Here $\epsilon(t) = \mathrm{sgn}\pi(it)$, $p(x) = \det^{1/2}$ $\{\sinh(\mathrm{ad}(x/2)/ \mathrm{ad}(x/2)\}$ and $\theta_t = F_{\mathfrak{g}}\mu_t$, as before.[For the definition of Θ_τ see [14] Theorem 1, p. 244 and Theorem 8, p.443. I have written τ for Varadarajan's b^*, Θ_τ for his $\Theta_{\omega(b^*)}$ and used the fact that $\tau(\exp(s)) = \deg(\tau)\, e^{(it-r,s)}$. (16) is a special case of equation (14) in Lemma 7, p.248 of [14], namely the case $b = 1$ in T and $\Omega^{(b)} = \exp(\mathfrak{g}^0)$.]
The relation (16) is essentially Kirillov's formula for the discrete series characters. To see this we need to recall the construction of Kirillov's canonical measure μ_Ω on an orbit Ω. First define a skew form B_x on the tangent space $T_x\Omega = \mathrm{ad}(\mathfrak{g})x$ for each x in Ω by the formula $B_x(u, v) = (x, [y, z])$ if $u = \mathrm{ad}(y)x$ and $v = \mathrm{ad}(z)x$ for some y, z in $\mathfrak{g}$. As this from is this form is non degenerate one can define a volume element in $T_x\Omega$ in the usual way: assign the volume $|\det B_x(u_i, u_j)|^{1/2}$ to the parallelepiped spanned by a basis $\{u_i\}$ of $T_x\Omega$. Having defined a volume element in each tangent space, we get a smooth measure on Ω, which is easily seen to be G-invariant. For reasons which will become clear shortly we multiply this measure by the constant $(2\pi)^{-\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{t})}$ to arrive at Kirillov's measure μ_Ω.
On the other band, for t in $\mathfrak{t}_r$ we have the G-invariant measure ν_t on $G\cdot t$ defined by

$$\int_{G\cdot t} f(x)\, d\nu_t(x) = \int_G f(g\cdot t)\, dg.$$

To compare ν_t with μ_t we note that the invertible analytic map $G/T\times\mathfrak{t}_+ \to \mathfrak{g}_r$ ($\mathfrak{t}_+ \subset \mathfrak{t}_r$ a fundamental domain for W) transforms the measures according to the formula

$$\int_{\mathfrak{g}_e} f(x)\, dx = |W| \int_{\mathfrak{t}_r} \int_{G\cdot t} f(y) d\nu_t(y)\, |\pi(t)|^2\, dt.$$

So if we write $d_t\nu_t$ for the volume element of ν_t in the tangent space $\mathrm{ad}(\mathfrak{g})t = \mathfrak{t}^\perp$ of Ω at t ($\mathfrak{t}^\perp$ the subspace of $\mathfrak{g}$ orthogonal orthogonal to $\mathfrak{t}$ with respect to (,), then the decomposition $\mathfrak{g} = \mathfrak{t}^\perp + \mathfrak{t}$ of $\mathfrak{g}$ corresponds to the decomposition $dx = d\nu_t(y)\, |\pi(t)|^2\, dt$ of dx. Thus $d\nu_t$ differs from the volume element of the metric (,) by the factor $(2\pi)^{-\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{t})}|W|^{-1}|\pi(t)|^{-2}$. (The factor $(2\pi)^{-\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{t})}$ comes from the normalizations of dx and dt in terms of Fourier transforms) This means that $d_t\nu_t$ assigns the volume

$$(2\pi)^{-\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{t})}|W|^{-1}|\pi(t)|^{-2}|\det(x_i, x_j)|^{\frac{1}{2}}$$

to the parallelepiped spanned by a basis $\{x_i\}$ in $\mathfrak{t}^\perp$. Comparing this with the volume

$$\begin{aligned}&(2\pi)^{-\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{t})}|\det B(x_i, x_j)|^{\frac{1}{2}}\\ &\quad= (2\pi)^{-\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{t})}|\det(t,\ [\mathrm{ad}_{\mathfrak{t}^\perp}(t)^{-1}x_i, \mathrm{ad}_{\mathfrak{t}^\perp}(t)^{-1}x_j)]|^{\frac{1}{2}}\\ &\quad= (2\pi)^{-\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{t})}|\pi(t)\backslash^{-1}|\det(t,\ [x_i, x_j])|^{\frac{1}{2}}\end{aligned}$$

assigned to this parallelepiped by the volume element μ_Ω, we get that

$$\begin{aligned}\mu_\Omega &= |W|\, |\pi(t)|\, \nu_t\\ &= |W|\, \epsilon(t)\, (i)^{-\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{t})}\mu,\end{aligned}$$

where μ is defined by (5) and $\epsilon(t) = \mathrm{sgn}\,\pi(it)$ as before (so that $|\pi(t)| = |\pi(it)| = \epsilon(t)\pi(it) = \epsilon(t)i)^{-\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{t})}\pi(t)$.) Substituting into(16) using $\theta_t = F_{\mathfrak{g}}\mu_t$ we find that $\Theta_\tau = \deg(\tau)\, F_{\mathfrak{g}}\mu_\Omega,$. So if we write π for the discrete series representation with character Θ_τ and use the notation introduced above we get the

Character formula for the discrete series of groups of type $\mathcal{H}$:

$$\mathrm{tr} \int_{\mathfrak{g}} \varphi(x)\, \pi(\exp(x))\, dx = \deg(\tau) \int_\Omega \{\int_{\mathfrak{g}} e^{i(\lambda,x)}\, \varphi(x)\, p(x)^{-1}\}\, d\mu_\Omega(\lambda).$$

for all C^∞ functions φ with compact support in $\mathfrak{g}^0$.

One should note that this formula determines the characters Θ_τ only on the open subset $\exp(\mathfrak{g}^0)$ of G. In fact it may well happen that different irreducible characters τ of T have the same differential, so that different Θ_τ's may cor respond to the same θ_t. If G is connected, however, then so is T and this situation cannot arise.

Note also that the formula differs from (Φ) in the introduction by the factor $\deg(\tau)$. This kind of extra factor first appeared in a paper of Kalgui [Comptes Rendus 284, (1977), p531].

We now turn to the other characters which occur in the Plancherel formula. From Harish-Chandra's work one knows that the irreducible representations of

G which occur, in the Plancherel formula are induced from cuspidal parabolic subgroups [8]: If P is a cuspidal parabolic in G with Langlands decomposition $P = MAN$, choose a discrete series representation σ of M and a regular unitary character ν of A to define a representation $\pi_{\sigma,\nu}$ of P by setting $\pi_{\sigma,\nu}\ (man) = \sigma(m)\nu(a)$. Let T be a compact Cartan subgroup of M, τ the regular, irreducible character of T which parametrizes the discrete series representation σ of M. Let t be the element in $\mathfrak{t}_r$ so that $s \mapsto \deg(\tau)(it - r, s)$ is the differential of τ and let a be the element in $\mathfrak{a}$ so that $b \to i(a, b)$ is the differential of ν.
(Here (,) is the Killing form of $\mathfrak{g}$.) *Let* $\Omega = \Omega(\sigma)$ be the G-orbit of $t + a$ in $\mathfrak{g}$, μ_Ω the canonical G-invariant measure on Ω. Finally let $\pi = \pi(\sigma, \nu)$ be the representation of G unitarily induced from the representation $\pi_{\sigma,\nu}$ of P. With this notation we have the

Character formula for principal series of groups of type $\mathcal{H}$:

$$\operatorname{tr} \int_{\mathfrak{g}} \varphi(x)\, \pi(\exp(x))\, dx = \deg(\tau) \int_\Omega \{ \int_{\mathfrak{g}} e^{i(\lambda,x)}\, \varphi(x)\, p(x)^{-1} \}\, d\mu_\Omega(\lambda).$$

for all C^∞ functions φ with compact support in $\mathfrak{g}^0$.

When P is a minimal parabolic (i.e. when M is compact) this bas been proved by Duflo by a reduction to Kirillov's formula for compact groups [3]. The proof for an arbitrary cuspidal parabolic is an entirely analogous reduction to Kirillov's formula for the discrete series characters of groups of class $\mathcal{H}$ established above. [Duflo assumes implicitly that G is linear, which allows him to drop the factor $\deg(\tau)$ from the formula. He also uses linearity in the proof of Lemma 2 in [2], but that lemma is valid for all groups of type $\mathcal{H}$, as one can verify using [14] Proposition 4, p. 193.]

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