Seminar on linear algebraic groups

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Wednesdays 8:30-10:00 am, M009

Contents

In this seminar, we will learn the basic theory of algebraic groups over any algebraic closed field. More precisely, we shall concentrate on affine algebraic groups, which are closed subvarieties of the general linear group $GL(n)$.

Affine algebraic groups play an important role in many areas of mathematics especially in geometry and representation theory. We shall deal with the following notations:

- the Lie algebra of an algebraic groups
- quotient of an algebraic group by a closed subgroup
- tori, solvable groups and Borel subgroups
- structure of reductive groups
- classification of semisimple algebraic groups
- representations of semisimple algebraic groups

In this seminar we mainly follow Borel's Book [Bo91]. There are many books and notes on this topic, you are free to choose references to cover the corresponding contents in your talk.

Program of the talk

Talk 1(18.10.2017): Overview.

Talk 2(25.10.2017): Background on algebraic geometry.
References: [Bo91, Chapter AG].

Quickly review definitions and results in §5-§10 on classical algebraic geometry, and §16-§17 on tangent spaces and simple points.

Talk 3(8.11.2017): First properties of algebraic groups .

Reference: [Bo91, Chapter I, §1-§2].

Define algebraic groups and action on a variety, then prove that any affine algebraic groups are linear([Bo91, Prop. 1.10]). Define group closure, solvable and nilpotent groups and prove basic properties in §2.

Talk 4(15.11.2017): Lie algebra.

Reference: [Bo91, Chapter I, §3].

Define the Lie algebra of an algebraic group and identify the Lie algebra of a closed subgroup $H \subset G$ as a subalgebra of Lie(G). Briefly review §3.16-3.22 if time allows.

Talk 5(22.11.2017): Jordan decomposition.

Reference: [Bo91, Chapter I, §4]

Talk 6(29.11.2017): Homogeneous spaces.

Reference: [Bo91, Chapter II §5-§7].

Prove Chevalley's semi-invariant theorem(Thm 5.1) and define semi-invariants, then use this to construct quotients of linear groups by closed subgroups. Skip most proofs.

Talk 7(6.12.2017): Diagonalizable groups and Tori.

Reference: [Bo91, Chapter III §8].

Define diagonalizable groups, weights and roots, quotients by tori, and prove basic properties.

Talk 8(13.12.2017): Conjuagacy classes and centralizers of semi-simple elements.

Reference: [Bo91, Chapter III §9].

Prove that conjugacy classes of semi-simple elements are closed [Bo91, Theorem 9.2].

Talk 9(20.12.2017): Connected solvable groups.

Reference: [Bo91, Chapter. III §10].

Prove the Lie-Kolchin Theorem ([Bo91, Corollary 10.5]) and the structure theorem of connected solvable groups ([Bo91, Theorem 10.6]). Show that one dimensional groups are isomorphic to $\mathbb{G}_a$ or $\mathbb{G}_m$.

Talk 10(10.1.2018): Borel subgroups.

Reference: [Bo91, Chapter IV §11].

Define Borel subgroups, conjugacy of Borel subgroups, Cartan subgroups, maximal tori, Weyl groups, radical, reductive and semi-simple groups. Prove a theorem of Chevalley ([Bo91, Theorem 11.16])

Talk 11(17.1.2018): Roots of a reductive group.

Reference: [Bo91, Chapter IV §12-§13].

Study further properties of Cartan subgroups, regular and singular tori, groups of semi-simple rank one. Prove the theorem on roots of a reductive group ([Bo91, Theorem 13.18]).

Talk 12(24.1.2018): Abstract root systems.

Reference: [Bo91, Chapter IV §14.1-14.8].

Explain abstract root systems and show how a reductive group (over an algebraic closed field) gives rise to a root system.

Talk 13(31.1.2018): Bruhat decomposition.

Reference: [Bo91, Chapter IV §14.9-14.26].

Prove the Bruhat decomposition ([Bo91, Theorem 14.12]). Study properties of standard parabolic subgroups and Levi subgroups.

References

[Bo91] A. Borel, *Linear algebraic groups*, GTM Vol. 126, Springer-Verlag, 1991.

[Ha77] R. Hartshorne, *Algebraic geomerty*, Graduate texts in mathematics 52, Springer, 1977.

[Hu75] J. Humphreys, *Linear algebraic groups*, GTM Vol. 21, Springer-Verlag, 1975.

[Pe] N. Perrin, *Linear algebraic groups*, lecture notes, available at `http://www.hcm.uni-bonn.de/homepages/prof-dr-nicolas-perrin/teaching/linear-algebraic-groups/`.

[Sp81] T.A. Springer, *Linear algebraic groups*, Progress in Mathematics, 9. Birkhuser, Boston, Mass., 1981.

Linear algebraic groups. s. If Q = C, every affine algebraic group G can be viewed as a complex Lie group; then G is connected as an algeqraic group, if and only if G is connected as a Lie group.Â algebraic set, G operates morphically on V (or G is an algebraic transformation group) when there is given a morphism -r: G x V -+ V with the usual properties of transformation groups. It operates k-morphically if G, V and -r are defined over k. An elementary, but basic, property of algebraic transformation groups is the existence of at least one closed orbit (e.g. an orbit of smallest possible dimension [I, Â§16)). (1) Special Linear Groups (2) Orthogonal Groups of odd dimension (3) Symplectic Groups (4) Orthogonal Groups of even dimension (5) Unitary Groups. 1.2. Representation theory. Let G be a group. While working with some examples of groups we quickly realise that there is not much we know about them. One way is possibly to compare them with some known groups. In this case we take the known groups as linear group, GL(V), and the comparison is made via all group homomorphisms from G to linear groups. More precisely, we dene a representation.Â In the theory of Linear algebraic groups (over k = kÂ¯) we associate root datum to a reductive group. In general root datum is more general than root system. 4.1. Root Datum vs Root System Definition 4.1.1 (Root Datum). Introduction. Algebraic group: a group that is also an algebraic variety such that the group operations are maps of varieties. Example. G = GLn(k), k = k. Goal: to understand the structure of reductive/semisimple ane algebraic groups over algebraically closed elds k (not necessarily of characteristic 0). Roughly, they are classied by their Dynkin diagrams, which are associated graphs. Within G are maximal, connected, solvable subgroups, called the Borel subgroups. Example. In G = GLn(k), a Borel subgroup B is given by the upper triangular matrices. A fundamental fact is that the Borels are con