CURRICULUM HANDBOOK

Mathematics

Chapter 6. The Future of Mathematics Education

by Lynn Arthur Steen

Pattern

“He just saw further than the rest of us.” The subject of this remark, cyberneticist Norbert Wiener, is one of many exceptional scientists who broke the bonds of tradition to create entirely new domains for mathematicians to explore. Seeing and revealing hidden patterns are what mathematicians do best. Each major discovery opens new areas rich with potential for further exploration. In the last century alone, the number of mathematical disciplines has grown at an exponential rate; examples include the ideas of Georg Cantor on transfinite sets, Sonja Kovalevsky on differential equations, Alan Turing on computability, Emmy Noether on abstract algebra, and, most recently, Benoit Mandelbrot on fractals.

To the public these new domains of mathematics are *terra incognita*. Mathematics, in the common lay view, is a static discipline based on formulas taught in the school subjects of arithmetic, geometry, algebra, and calculus. But outside public view, mathematics continues to grow at a rapid rate, spreading into new fields and spawning new applications. The guide to this growth is not calculation and formulas but an open-ended search for pattern.

Mathematics has traditionally been described as the science of number and shape. The school emphasis on arithmetic and geometry is deeply rooted in this centuries-old perspective. But as the territory explored by mathematicians has expanded—into group theory and statistics, into optimization and control theory—the historic boundaries of mathematics have all but disappeared. So have the boundaries of its applications: no longer just the language of physics and engineering, mathematics is now an essential tool for banking, manufacturing, social science, and medicine. When viewed in this broader context, we see that mathematics is not just about number and shape but about pattern and order of all sorts. Number and shape—arithmetic and geometry—are but two of many media in which mathematicians work. Active mathematicians seek patterns wherever they arise.

Thanks to computer graphics, much of the mathematician's search for patterns is now guided by what one can really see with the eye, whereas nineteenth-century mathematical giants like Gauss and Poincaré had to depend more on seeing with their mind's eye. “I see” has always had two distinct meanings: to perceive with the eye and to understand with the mind. For centuries the mind has dominated the eye in the hierarchy of mathematical practice; today the balance is being restored as mathematicians find new ways to see patterns, both with the eye and with the mind.

Change in the practice of mathematics forces re-examination of mathematics education. Not just computers, but also new applications and new theories have significantly expanded the role of mathematics in science, business, and technology. Students who will live and work using computers as a routine tool need to learn a different mathematics than their ancestors. Standard school practice, rooted in traditions that are several centuries old, simply cannot prepare students adequately for the mathematical needs of the 21st century.

Shortcomings in the present record of mathematical education also provide strong forces for change. Indeed, because new developments build on fundamental principles, it is plausible, as many observers often suggest, that one should focus first on restoring strength to time-honored fundamentals before embarking on reforms based on changes in the contemporary practice of mathematics. Public support for strong basic curricula reinforces the wisdom of the past—that traditional school mathematics, if carefully taught and well learned, provides sound preparation both for the world of work and for advanced study in mathematically based fields.

The key issue for mathematics education is not *whether* to teach fundamentals but *which* fundamentals to teach and *how* to teach them. Changes in the practice of mathematics do alter the balance of priorities among the many topics that are important for numeracy. Changes in society, in technology, in schools—among others—will have great impact on what will be possible in school mathematics in the next century. All of these changes will affect the fundamentals of school mathematics.

To develop effective new mathematics curricula, one must attempt to foresee the mathematical needs of tomorrow's students. It is the present and future practice of mathematics—at work, in science, and in research—that should shape education in mathematics. To prepare effective mathematics curricula for the future, we must look to patterns in the mathematics of today to project, as best we can, just what is and what is not truly fundamental.

Fundamental Mathematics

School tradition has it that arithmetic, measurement, algebra, and a smattering of geometry represent the fundamentals of mathematics. But there is much more to the root system of mathematics—deep ideas that nourish the growing branches of mathematics. One can think of specific mathematical

- **structures:**
- numbers
- algorithms
- ratios
- shapes
- functions
- data
- **or attributes:**
- linear
- periodic
- symmetric
- continuous
- random
- maximum
- approximate

- smooth
- **or actions:**
- represent
- control
- prove
- discover
- apply
- model
- experiment
- classify
- visualize
- compute
- **or abstractions:**
- symbols
- infinity
- optimization
- logic
- equivalence
- change
- similarity
- recursion
- **or attitudes:**
- wonder
- meaning
- beauty
- reality
- **or behaviors:**
- motion
- chaos
- resonance
- iteration
- stability
- convergence
- bifurcation
- oscillation
- **or dichotomies:**
- discrete vs. continuous
- finite vs. infinite
- algorithmic vs. existential
- stochastic vs. deterministic
- exact vs. approximate

These diverse perspectives illustrate the complexity of structures that support mathematics. From each perspective one can identify various strands that have within them the power to develop a significant mathematical idea from informal intuitions of early childhood all the way through school and college and on into scientific or mathematical research. A sound education in the mathematical sciences requires encountering virtually all of these very different perspectives and ideas.

Traditional school mathematics picks very few strands (e.g., arithmetic, geometry, algebra) and arranges them horizontally to form the curriculum: first arithmetic, then simple algebra, then geometry, then more algebra, and finally—as if it were the epitome of mathematical knowledge—calculus. This layer-cake approach to mathematics education effectively prevents informal development of intuition along the multiple roots of mathematics. Moreover, it reinforces the tendency to design each course primarily to meet the prerequisites of the next course, making the study of mathematics largely an exercise in delayed gratification. To help students see clearly into their own mathematical futures, we need to construct curricula with greater vertical continuity, to connect the roots of mathematics to the branches of mathematics in the educational experiences of children.

School mathematics is often viewed as a pipeline for human resources that flows from childhood experiences to scientific careers. The layers in the mathematics curriculum correspond to increasingly constricted sections of pipe through which all students must pass if they are to progress in their mathematical and scientific education. Any impediment to learning, of which there are many, restricts the flow in the entire pipeline. Like cholesterol in the blood, poor mathematics instruction can clog the educational arteries of the nation.

In contrast, if mathematics curricula featured multiple parallel strands, each grounded in appropriate childhood experiences, the flow of human resources would more resemble the movement of nutrients in the roots of a mighty tree—or the rushing flow of water from a vast watershed—than the increasingly constricted confines of a narrowing artery or pipeline. Different aspects of mathematical experience will attract children of different interests and talents, each nurtured by challenging ideas that stimulate imagination and promote exploration. The collective effect will be to develop among children diverse mathematical insight in many different roots of mathematics.

Five Samples

This volume offers five examples of the developmental power of deep mathematical ideas: dimension, quantity, uncertainty, shape, and change. Each chapter explores a rich variety of patterns that can be introduced to children at various stages of school, especially at the youngest ages when unfettered curiosity remains high. Those who develop curricula will find in these essays many valuable new options for school mathematics. Those who help determine education policy will see in these essays examples of new standards for excellence. And everyone who is a parent will find in these essays numerous examples of important and effective mathematics that could excite the imaginations of their children.

Each chapter is written by a distinguished scholar who explains in everyday language how fundamental ideas with deep roots in the mathematical sciences could blossom in schools. Although not constrained by particular details of present curricula, each essay is faithful to the development of mathematical ideas from childhood to adulthood. In expressing these very different strands of mathematical thought, the authors illustrate ideals of how mathematical ideas should be developed in children.

In contrast to much present school mathematics, these strands are alive with action: pouring water to compare volumes, playing with pendulums to explore dynamics, counting candy colors to grasp variation, building kaleidoscopes to explore symmetry. Much mathematics can be learned informally by such activities long before children reach the point of understanding algebraic formulas. Early experiences with such patterns as volume, similarity, size, and randomness prepare students both for scientific investigations and for more formal and logically precise mathematics. Then when a careful demonstration emerges in class some years later, a student who has benefited from substantial early informal mathematical experiences can say with honest pleasure, “Now I see why that's true.”

Connections

The essays in *On the Shoulders of Giants* were written by five different authors on five distinct topics. Despite differences in topic, style, and approach, these essays have in common the lineage of mathematics: each is connected in myriad ways to the family of mathematical sciences. Thus it should come as no surprise that the essays themselves are replete with interconnections, both in deep structure and even in particular illustrations. Some examples:

- **Measurement** is an idea treated repeatedly in these essays. Experience with geometric quantities (length, area, volume), with arithmetic quantities (size, order, labels), with random variation (spinners, coin tosses, SAT scores), and with dynamic variables (discrete, continuous, chaotic) all pose special challenges to answer a very child-like question: “How big is it?” One sees from many examples that this question is fundamental: it is at once simple yet subtle, elementary yet difficult. Students who grow up recognizing the complexity of measurement may be less likely to accept unquestioningly many of the common misuses of numbers and statistics. Learning how to measure is the beginning of numeracy.
- **Symmetry** is another deep idea of mathematics that turns up over and over again, both in these essays and in all parts of mathematics. Sometimes it is the symmetry of the whole, such as the hypercube (a four-dimensional cube), whose symmetries are so numerous that it is hard to count them all. (But with proper guidance, young children using a simple pea-and-toothpick model can do it.) Other times it is the symmetry of the parts, as in the growth of natural objects from repetitive patterns of molecules or cells. In still other cases it is symmetry broken, as in the buckling of a cylindrical beam or the growth of a fertilized egg to a slightly asymmetrical adult animal. Unlike measurement, symmetry is seldom studied much in school at any level, yet it is equally fundamental as a model for explaining features of such diverse phenomena as the basic forces of nature, the structure of crystals, and the growth of organisms. Learning to recognize symmetry trains the mathematical eye.
- **Visualization** recurs in many examples in this volume and is one of the most rapidly growing areas of mathematical and scientific research. The first step in data analysis is the visual display of data to search for hidden patterns. Graphs of various types provide visual display of relations and functions; they are widely used throughout science and industry to portray the behavior of one variable (e.g., sales) that is a function of another (e.g., advertising). For centuries, artists and map makers have used geometric devices such a projection to represent three-dimensional scenes on a two-dimensional canvas or sheet of paper. Now, computer graphics automate these processes and let us explore as well the projections of shapes in higher-dimensional space. Learning to visualize mathematical patterns enlists the gift of sight as an invaluable ally in mathematical education.
- **Algorithms** are recipes for computation that occur in every corner of mathematics. A common iterative procedure for projecting population growth reveals how simple orderly events can lead to a variety of behaviors—explosion, decay, repetition, and chaos. Exploration of combinatorial patterns in geometric forms enables students to project geometric structures in higher dimensions where they cannot build real models. Even common elementary school algorithms for arithmetic take on a new dimension when viewed from the perspective of contemporary mathematics. Rather than stressing the mastery of specific algorithms— which are now carried out principally by calculators or computers—school mathematics can instead emphasize more fundamental attributes of algorithms (e.g., speed, efficiency, sensitivity) that are essential for intelligent use of mathematics in the computer age. Learning to think algorithmically builds contemporary mathematical literacy.

Many other connective themes recur in this volume, including linkages of mathematics with science, classification as a tool for understanding, inference from axioms and data, and—most importantly—the role of exploration in the process of learning mathematics. Connections give mathematics power and help determine what is fundamental. Pedagogically, connections permit insight developed in one strand to infuse into others. Multiple strands linked by strong interconnections can develop mathematical power in students with a wide variety of enthusiasms and abilities.

Gaining Perspective

Newton credited his extraordinary foresight in the development of calculus to the accumulated work of his predecessors: “If I have seen farther than others, it is because I have stood on the shoulders of giants.” Those who develop mathematics curricula for the 21st century will need similar foresight.

Not since the time of Newton has mathematics changed as much as it has in recent years. Motivated in large part by the introduction of computers, the nature and practice of mathematics have been fundamentally transformed by new concepts, tools, applications, and methods. Like the telescope of Galileo's era that enabled the Newtonian revolution, today's computer challenges traditional views and forces re-examination of deeply held values. As it did three centuries ago in the transition from Euclidean proofs to Newtonian analysis, mathematics once again is undergoing a fundamental reorientation of procedural paradigms.

Examples of fundamental change abound in the research literature of mathematics and in practical applications of mathematical methods. Many are given in the essays in this volume:

- Uncertainty is not haphazard, since regularity eventually emerges.
- Deterministic phenomena often exhibit random behavior.
- Dimensionality is not just a property of space but also a means of ordering knowledge.
- Repetition can be the source of accuracy, symmetry, or chaos.
- Visual representation yields insights that often remain hidden from strictly analytic approaches.
- Diverse patterns of change exhibit significant underlying regularity.

By examining many different strands of mathematics, we gain perspective on common features and dominant ideas. Recurring concepts (e.g., number, function, algorithm) call attention to what one must know in order to *understand* mathematics. Common actions (e.g., represent, discover, prove) reveal skills that one must develop in order to *do* mathematics. Together, concepts and actions are the nouns and verbs of the language of mathematics.

What humans do with the language of mathematics is to describe patterns. Mathematics is an exploratory science that seeks to understand every kind of pattern—patterns that occur in nature, patterns invented by the human mind, and even patterns created by other patterns. To grow mathematically, children must be exposed to a rich variety of patterns appropriate to their own lives through which they can see variety, regularity, and interconnections.

The essays in this volume (*On the Shoulders of Giants*) provide five extended case studies that exemplify how this can be done. Other authors could just as easily have described five or ten different examples. The books and articles listed below are replete with additional examples of rich mathematical ideas. What matters in the study of mathematics is not so much which particular strands one explores, but the presence in these strands of significant examples of sufficient variety and depth to reveal patterns. By encouraging students to explore patterns that have proven their power and significance, we offer them broad shoulders from which they will see farther than we can.

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The key issue for mathematics education is not whether to teach fundamentals but which fundamentals to teach and how to teach them. Changes in the practice of mathematics do alter the balance of priorities among the many topics that are important for numeracy. Changes in society, in technology, in schoolsâ€”among othersâ€”will have great impact on what will be possible in school mathematics in the next century.Â To prepare effective mathematics curricula for the future, we must look to patterns in the mathematics of today to project, as best we can, just what is and what is not truly fundamental. Fundamental Mathematics. School tradition has it that arithmetic, measurement, algebra, and a smattering of geometry represent the fundamentals of mathematics. The progression of both the nature of mathematics and individual mathematical problems into the future is a widely debated topic - many past predictions about modern mathematics have been misplaced or completely false, so there is reason to believe that many predictions today may follow a similar path. However, the subject still carries an important weight and has been written about by many notable mathematicians. Typically, they are motivated by a desire to set a research agenda to direct efforts to